COMBINATORIAL MODELS OF EXPANDING DYNAMICAL SYSTEMS

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Contents

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Abstract. We define iterated monodromy groups of more general structures than partial self-covering. This generalization makes it possible to define a natural notion of a combinatorial model of an expanding dynamical system. We prove that a naturally defined "Julia set" of the generalized dynamical systems depends only on the associated iterated monodromy group. We show then that the Julia set of every expanding dynamical system is an inverse limit of simplicial complexes constructed by inductive cut-and-paste rules.

1. Introduction

According to a well known principle, expanding (and more generally, hyperbolic) dynamical systems have combinatorial nature and are determined by a finite amount of data. For instance, they are *finitely presented*, see [Fri87]. This principle can be also formulated in a form of structural stability or rigidity theorems: two hyperbolic dynamical systems that are topologically or homotopically close to each other are conjugate.

The aim of our paper is to describe, by proving the corresponding rigidity theorem, a complete algebraic invariant of expanding dynamical systems. We translate then this algebraic invariant into a more geometric language of polyhedral models of dynamical systems (and their Julia sets). These models give a representation of the Julia set of the dynamical system as an inverse limit of simplicial complexes that are constructed using simple cut-and-paste rules, similar to subdivision rules in one-dimensional complex dynamics. We illustrate our techniques, in particular, by constructing combinatorial models of the Julia sets of multi-dimensional dynamical systems.

We define the algebraic invariant (called the *iterated monodromy group*) in a general setting of a multi-valued partially defined dynamical system. Namely, a *topological automaton* is a pair of maps $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$, $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ between two topological spaces (or orbispaces), such that f is a finite covering map. If ι is a homeomorphism, then we can identify $\mathcal{M}_1$ and $\mathcal{M}$ by ι, thus getting a dynamical system $f : \mathcal{M} \longrightarrow \mathcal{M}$. If ι is an embedding, then $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a partial self-covering.

Iterated monodromy groups were originally defined for partial self-coverings only (see [BGN03, Nek05]). However, the fact that ι is an embedding is not used neither in the construction nor in the main results of [Nek05]. Moreover, iterated monodromy groups of partial self-coverings of orbispaces are defined in [Nek05] essentially in the setting of topological automata.

Topological automata were studied (under different names) by T. Katsura in [Kat04] in relation with C^*-algebras, and by Y. Ishii and J. Smillie [IS08] in relation with homotopical rigidity of hyperbolic dynamical systems. The last article was one of inspirations of our paper.

A topological automaton $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ can be naturally iterated. The covering f and the map ι induce a covering $f_1 : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ and a map $\iota_1 : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ defined by the pull-back diagram

$$\begin{array}{ccc} \mathcal{M}_2 & \xrightarrow{\iota_1} & \mathcal{M}_1 \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{M}_1 & \xrightarrow{\iota} & \mathcal{M} \end{array}$$

We define then, inductively, coverings $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ and maps $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$. The nth iteration of the pair $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is then the pair

$$f \circ f_1 \circ \cdots \circ f_{n-1}, \quad \iota \circ \iota_1 \circ \cdots \circ \iota_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}.$$

If ι is an embedding, then the spaces $\mathcal{M}_n$ are the domains of the iterations f^n of the partial map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$.

The iterated monodromy of a topological automaton is defined in Section 4. Rather than to give the definition here, we define an equivalent notion of the associated virtual endomorphism of the fundamental group. Suppose that the space $\mathcal{M}$ is path connected and locally path connected. Since $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a finite covering map, the induced map $f_* : \pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$ is an embedding, and $f_*(\pi_1(\mathcal{M}_1))$ has finite index in $\pi_1(\mathcal{M})$. The *virtual endomorphism* associated with the topological automaton $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is the homomorphism $\iota_* \circ f_*^{-1}$ from the subgroup $f_*(\pi_1(\mathcal{M}_1)) \le \pi_1(\mathcal{M})$ to $\pi_1(\mathcal{M})$. It is well defined up to inner automorphisms of $\pi_1(\mathcal{M})$.

If two topological automata $f', \iota' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ and $f'', \iota'' : \mathcal{M}''_1 \longrightarrow \mathcal{M}''$ have the same associated virtual endomorphisms ϕ' and ϕ'' (i.e., if there exists an isomorphism $\alpha : \pi_1(\mathcal{M}') \longrightarrow \pi_1(\mathcal{M}'')$ such that $\alpha \circ \phi'$ is equal to $\phi'' \circ \alpha$ modulo inner automorphisms), then the topological automata are called *combinatorially equivalent*. More generally, the automata are combinatorially equivalent, if we can make the associated virtual endomorphisms the same by taking quotients of the fundamental groups by normal subgroups invariant under the action of the virtual endomorphisms. A more precise definition is given in Subsection 4.6.

A topological automaton $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is called *contracting* if $\mathcal{M}$ and $\mathcal{M}_1$ are compact length spaces (e.g., Riemannian manifolds, or simplicial complexes with Riemannian structure on simplices), f is a local isometry, and ι is contracting. If ι is a homeomorphism or an embedding, then it could be more natural to restrict the length structure of $\mathcal{M}$ onto $\mathcal{M}_1$. In this setting an equivalent condition is that f is *expanding*.

Our first main result is the following rigidity theorem (see Theorems 5.9 and 5.10).

Theorem 1.1. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a contracting topological automaton with semi-locally simply connected space $\mathcal{M}$. Denote by $\lim_\iota \mathcal{F}$ the inverse limit of the sequence of spaces and maps*

$$\mathcal{M} \xleftarrow{\iota} \mathcal{M}_1 \xleftarrow{\iota_1} \mathcal{M}_2 \xleftarrow{\iota_2} \mathcal{M}_3 \xleftarrow{\iota_3} \cdots .$$

Let $f_\infty : \lim_\iota \mathcal{F} \longrightarrow \lim_\iota \mathcal{F}$ be the map induced by the coverings f_n. Then the dynamical system $(\lim_\iota \mathcal{F}, f_\infty)$ depends, up to topological conjugacy, only on the combinatorial equivalence class of the topological automaton $\mathcal{F}$.

In fact, we prove that the dynamical system $(\lim_\iota \mathcal{F}, f_\infty)$ is topologically conjugate with the *limit dynamical system* of the iterated monodromy group of $\mathcal{F}$. Limit dynamical systems of contracting self-similar groups (contracting virtual endomorphisms) were defined in [BGN03, Nek05] using symbolic dynamics (as quotients of the space of infinite sequences by an equivalence relation defined by a group action).

The inverse limit $\lim_\iota \mathcal{F}$ is an analogue of the Julia set of an expanding dynamical system. For example, suppose that $f \in \mathbb{C}(z)$ is a hyperbolic rational function of one complex variable acting on the Riemann sphere. Then there exists a compact neighborhood $\mathcal{M}$ of the Julia set of f such that $f^{-1}(\mathcal{M}) \subset \mathcal{M}$ and $\mathcal{M}$ does not contain the critical values of f. Consider the topological automaton $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$,

where $\mathcal{M}_1 = f^{-1}(\mathcal{M})$ and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is the identical embedding. Then the inverse limit $\lim_\iota \mathcal{M}_n = \bigcap_{n\ge 1} \mathcal{M}_n$ is the Julia set of f. The automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ is contracting with respect to the restriction onto $\mathcal{M}$ of the Poincaré metric on the sphere minus the post-critical set of f.

A partial case of Theorem 1.1 (when $\mathcal{M}$ is a Riemannian manifold and ι is a diffeomorphism) is the theorem of M. Shub on expanding endomorphisms of manifolds, see [Shu69, Shu70].

Theorem 1.1 can be used now to approximate dynamical systems (acting on their Julia sets) by topological automata. For instance, if $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an expanding partial self-covering, then we can replace $\mathcal{M}, \mathcal{M}_1$, f and the embedding $\mathcal{M}_1 \hookrightarrow \mathcal{M}$ by homotopically equivalent spaces and maps $f', \iota' : \mathcal{M}_1' \longrightarrow \mathcal{M}'$, thus getting a topological automaton $\mathcal{F}$ combinatorially equivalent to the partial self-covering f. If we find a length structure on $\mathcal{M}'$ such that $\iota' : \mathcal{M}_1' \longrightarrow \mathcal{M}'$ is contracting with respect to the lift of the length structure of $\mathcal{M}'$ to $\mathcal{M}_1'$ by f, then Theorem 1.1 implies that the dynamical system $(\lim_{\iota'} \mathcal{F}, f'_\infty)$ is topologically conjugate to the action of f on its Julia set. (Here the Julia set of an expanding map is defined as the limit set of inverse iterations.) In particular, the spaces $\mathcal{M}_n'$ approximate the Julia set of f. A known example of this approach are the classical Hubbard trees of post-critically finite polynomials (see [DH84, DH85]), which are constructed by retracting the *Thurston orbispace* of the polynomial onto a finite tree. Our method has no restrictions on dimension of the spaces, and can be applied to any expanding dynamical system. For example, we construct polyhedral models of the Julia sets of post-critically finite endomorphisms of $\mathbb{CP}^n$ coming from Teimüller theory of post-critically finite polynomials.

A natural question arises now in connection with Theorem 1.1. How to construct a simple contracting topological automaton (e.g., consisting of simplicial complexes $\mathcal{M}, \mathcal{M}_1$ and piecewise affine maps f, ι) with given iterated monodromy group (with given associated virtual endomorphism)? Such a construction will provide approximations of the Julia sets of expanding dynamical systems in a general and systematic way.

Let $\phi : G_1 \longrightarrow G$ be a surjective virtual endomorphism of a finitely generated group G. Suppose that $\mathcal{X}$ is a path connected metric space on which G acts by isometries properly and co-compactly. Then the identity map on $\mathcal{X}$ induces a covering $f : \mathcal{X}/G_1 \longrightarrow \mathcal{X}/G$ of the corresponding orbispaces (if the action of G on $\mathcal{X}$ is free, then f is a covering of topological spaces). Suppose that a map $F : \mathcal{X} \longrightarrow \mathcal{X}$ is such that

$$F(\xi \cdot g) = F(\xi) \cdot \phi(g) \tag{1}$$

for all $g \in G_1$ and $\xi \in \mathcal{X}$. Then F induces a continuous map (a morphism of orbispaces) $\iota : \mathcal{X}/G_1 \longrightarrow \mathcal{X}/G$. We get in this way a topological automaton $\mathcal{F} = (\mathcal{X}/G_1, \mathcal{X}/G, f, \iota)$. If $\mathcal{X}$ is simply connected, then $\pi_1(\mathcal{X}/G) = G$ and the virtual endomorphism associated with the constructed topological automaton is ϕ. In general, if $\widetilde{\phi}$ is the virtual endomorphism of $\pi_1(\mathcal{X}/G)$ associated with the automaton $\mathcal{F}$, then G is the quotient of $\pi_1(\mathcal{X}/G)$ by a normal subgroup invariant under $\widetilde{\phi}$, and ϕ is the virtual endomorphism induced by $\widetilde{\phi}$ on the quotient. Iteration of the automaton $\mathcal{F}$ produces the spaces $\mathcal{M}_n = \mathcal{X}/\operatorname{Dom}\phi^{\circ n}$. The maps $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ are the coverings induced by the inclusions $\operatorname{Dom}\phi^{\circ(n+1)} \le \operatorname{Dom}\phi^{\circ n}$; the maps $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ are induced by the map F.

It follows that the question of finding a contracting topological automaton with given iterated monodromy group is equivalent to the question of finding a proper co-compact G-space $\mathcal{X}$ and a contracting map $F : \mathcal{X} \longrightarrow \mathcal{X}$ satisfying (1).

The most natural proper co-compact G-space is the group G itself with respect to right translations. Choose a left coset transversal $\{r_1 = 1, r_2, \ldots, r_d\}$ for the subgroup G_1. Then we can define a map $F : G \longrightarrow G$ satisfying (1) by the formula

$$F(g) = \phi(r_i g),$$

where r_i is such that $r_i g \in G_1$. But we need to have a metric space $\mathcal{X}$ such that F is contracting. A standard approach in geometric group theory is to consider the Cayley, or Rips complex of G (see [Gro87]). If S is a finite generating set of G, then denote by $\Gamma(G, S)$ the simplicial complex with the set of vertices G in which a subset $A \subset G$ is a simplex if and only if $A \cdot g^{-1} \subset S$ for all $g \in A$. If S is invariant with respect to the map F, then we get a simplicial map $F : \Gamma(G, S) \longrightarrow \Gamma(G, S)$ satisfying (1).

It is proved in Subsection 6.3 (Theorem 6.6) that this natural construction works.

Theorem 1.2. *If $\phi : G_1 \longrightarrow G$ is a contracting virtual endomorphism (e.g., the virtual endomorphism associated with a contracting automaton), then there exist positive integers m and n such that the map $F^{\circ n} : \Gamma(G, S^m) \longrightarrow \Gamma(G, S^m)$ is homotopic through maps satisfying* (1) *to a contracting map.*

In this way we get for every contracting topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ (e.g., for every expanding partial self-covering) a contracting simplicial topological automaton combinatorially equivalent to some iteration of $\mathcal{F}$. The Julia set of $\mathcal{F}$ will be homeomorphic to the inverse limit of the simplicial complexes $\Gamma(G, S^m)/\operatorname{Dom}\phi^{\circ nk}$ as $k \to \infty$.

Note that every finite-dimensional compact metric space is an inverse limit of simplicial complexes, by an old theorem of P. Alexandroff [Ale29].

Theorem 6.6 proved in our paper is more explicit and "cleaner" than Theorem 1.2. In particular, we use a smaller simplicial complex than $\Gamma(G, S^m)$ (the map F is not surjective on $\Gamma(G, S^m)$, so we can pass to the intersection of domains of its iteration).

Due to combinatorial nature of the simplicial complex $\Gamma(G, S^m)$, the complexes $\Gamma(G, S^m)/\operatorname{Dom}\phi^{\circ n}$ are constructed using simple recursive cut-and-paste rules, described in Proposition 6.5.

The structure of the paper is as follows. The second section is an overview of the techniques of self-similar groups, virtual endomorphisms, and their limit spaces. All proofs can be found in the monograph [Nek05].

In Section 3 we define topological automata and describe some examples (Moore diagrams of finite automata, wreath recursions, partial self-coverings, post-critically finite rational functions, post-critically finite correspondences, bi-reversible automata, commensurizers of tree lattices, Thurston maps, and subdivision rules).

In Section 4 we show how topological automata are iterated; define the inverse limit $\lim_\iota \mathcal{F}$, and two other inverse limits $\lim_f \mathcal{F}$ and $\lim_{f,\iota} \mathcal{F}$; define iterated monodromy groups of topological automata; and show how they are computed as self-similar groups. At the end of the section we define the notion of combinatorial equivalence of topological automata.

Section 5 studies contracting topological automata. We pass to a more convenient setting of group actions on topological spaces. If $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ is a topological automaton, then passing to the universal covering $\mathcal{X}$ of $\mathcal{M}$ we get an action of $\pi_1(\mathcal{M})$ on $\mathcal{X}$, a subgroup $G_1 \cong \pi_1(\mathcal{M}_1)$ of $\pi_1(\mathcal{M})$, and a map $F : \mathcal{X} \longrightarrow \mathcal{X}$, which is a lift of the map ι to the universal covering. The map F satisfies the condition (1) for the virtual endomorphism ϕ associated with the topological automaton. We formalize such structures, and pass from the study of topological automata to the study of group actions and equivariant maps. We prove then results equivalent to Theorem 1.1: one is formulated in terms of group actions (Theorem 5.9), and the other in terms of topological automata (Theorem 5.10). We also show that a topological automaton homotopy equivalent to a contracting topological automaton can be made contracting, if we pass to its iteration (Corollary 5.13). This result can be used to construct contracting topological automata approximating an expanding dynamical system by passing to homotopy equivalent spaces and maps.

In Section 6 we show how to construct a contracting piecewise affine topological automaton starting from any contracting iterated monodromy group (i.e., starting from any contracting virtual endomorphism of a group). Our construction essentially coincides with the one described in Theorem 1.2 above. The only difference is that we pass to the smaller complex $\bigcap_{k\ge 1} F^k(\Gamma(G, S^m))$, which will not depend now on the choice of S and m (if m is big enough). We also describe recurrent cut-and-paste rules for constructing the simplicial complexes approximating the Julia set (Proposition 6.5).

The last section presents some examples of application of the developed technique. In particular, we show how the Hubbard trees fit into our theory, and describe polyhedral models of post-critically finite rational endomorphisms of complex projective spaces coming from Teichmüller theory of hyperbolic polynomials.

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2. Self-similar groups and their limit spaces

We give in this section a short overview of the main definitions and constructions of the theory of self-similar groups. For more details and proofs, see [Nek05, Nek08b].

2.1. Main definitions.

For a finite set X, we denote by $\mathsf{X}^* = \bigsqcup_{n\ge 0} \mathsf{X}^n$ the set of finite words over X, i.e., the free monoid generated by X.

Definition 1. A *faithful self-similar action* (G, X) is a faithful action of a group G on the set X^* such that for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ such that

$$g(xw) = g(x)h(w)$$

for all $w \in \mathsf{X}^*$.

Every self-similar action preserves the levels X^n of X^*. It follows from the definition that for every word $v \in \mathsf{X}^*$ and every $g \in G$ there exists $h \in G$ such that

$$g(vw) = g(v)h(w).$$

The element h is unique, by faithfulness of the action. We denote $h = g|_v$ and call h the *section* (or *restriction*) of g at v. We have the following properties of sections

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \quad (g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v. \tag{2}$$

Self-similar actions are usually described by the *associated wreath recursion*, which is the homomorphism $\varphi : G \longrightarrow \mathfrak{S}(\mathsf{X}) \wr G = \mathfrak{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ given by

$$\varphi(g) = \pi \cdot (g|_x)_{x\in\mathsf{X}},$$

where π is the permutation of $\mathsf{X} = \mathsf{X}^1 \subset \mathsf{X}^*$ defined by g.

For example, the transformation of $\{0, 1\}^*$ defined by the recursive rules

$$a(0w) = 1w, \quad a(1w) = 0a(w)$$

is defined in terms of the associated wreath recursion as

$$\varphi(a) = \sigma(1, a),$$

where $\sigma \in \mathfrak{S}(\{0, 1\})$ is the transposition and 1 on the right hand side of the equality is the trivial transformation. We will usually omit φ and write the last equality as $a = \sigma(1, a)$.

The elements of the wreath product $\mathfrak{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ are multiplied according to the rule

$$\pi_1(g_x)_{x\in\mathsf{X}} \cdot \pi_2(h_x)_{x\in\mathsf{X}} = \pi_1\pi_2(g_{\pi_2(x)}h_x)_{x\in\mathsf{X}}.$$

Note that we use left action in this formula.

The wreath recursion uniquely determines the associated self-similar action. The following proposition is proved in [Nek05, Proposition 2.3.4] (see also [Nek08b, Proposition 2.12]).

Proposition 2.1. *Let (G, X) be a self-similar action and let $\phi : G \longrightarrow \mathfrak{S}(\mathsf{X}) \wr G$ be the associated wreath recursion. For every element $h \in \mathfrak{S}(\mathsf{X}) \wr G$ the self-similar action associated with the wreath recursion $g \mapsto h^{-1}\phi(g)h$ is conjugate to the self-similar action (G, X).*

Definition 2. Two self-similar actions of a group G on X^* are said to be *equivalent* if the associated wreath recursions can be obtained from each other by taking composition with an inner automorphism of the group $\mathfrak{S}(\mathsf{X}) \wr G$.

An approach equivalent to wreath recursions, but in some sense more "coordinate-free", uses the notion of a *permutational bimodule*, which is defined as follows.

Definition 3. Let G be a group. A *permutational G-bimodule* is a set $\mathfrak{M}$ together with commuting left and right actions of G on it, i.e., with two maps $G \times \mathfrak{M} \longrightarrow \mathfrak{M} : (g, x) \mapsto g \cdot x$ and $\mathfrak{M} \times G : (x, g) \mapsto x \cdot g$ satisfying the following conditions

(1) $1 \cdot x = x$, $x \cdot 1 = x$ for all $x \in \mathfrak{M}$;
(2) $g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$ and $(x \cdot g_1) \cdot g_2 = x \cdot (g_1g_2)$ for all $g_1, g_2 \in G$ and $x \in \mathfrak{M}$;
(3) $g_1 \cdot (x \cdot g_2) = (g_1 \cdot x) \cdot g_2$ for all $g_1, g_2 \in G$.

Let (G, X) be a self-similar action. If we identify the letters of X with the transformations

$$x : v \mapsto xv$$

of X^*, then the condition

$$g(xw) = yh(w) \quad \forall w \in \mathsf{X}^*$$

is written as the equality

$$g \cdot x = y \cdot h$$

of compositions of transformations of X^*. It follows that the set $\mathsf{X} \cdot G$ of transformations $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$ is a G-bimodule with respect to pre- and post-compositions with the elements of G. The obtained bimodule is called the *associated bimodule* of the self-similar action (or the *self-similarity bimodule*). The left and right actions of G on the set $\mathsf{X} \cdot G$ are given then by the rules

$$h \cdot (x \cdot g) = h(x) \cdot (h|_x g), \quad (x \cdot g) \cdot h = x \cdot (gh).$$

The right action of G on $\mathsf{X} \cdot G$ is free (i.e., $x \cdot g = x$ implies $g = 1$) and has $|\mathsf{X}|$ orbits. We generalize these conditions in the following definition.

Definition 4. A *(d-fold) covering G-bimodule* is a permutational G-bimodule $\mathfrak{M}$ such that the right action of G on $\mathfrak{M}$ is free and has d orbits.

A transversal $\mathsf{X} \subset \mathfrak{M}$ of the right orbits, i.e., a set intersecting every orbit of the right action exactly once, is called a *basis* of the covering bimodule $\mathfrak{M}$.

Let $\mathfrak{M}$ be a d-fold covering G-bimodule. Choose a basis X. Then every element of $\mathfrak{M}$ can be uniquely written in the form $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$. Consequently, for every $g \in G$ and $x \in \mathsf{X}$ there exist unique $h \in G$ and $y \in \mathsf{X}$ such that $g \cdot x = y \cdot h$ in $\mathfrak{M}$. The *associated self-similar action* $(G, \mathsf{X}, \mathfrak{M})$ of G on X^* is given then by the recurrent rule

$$g(xw) = yh(w) \Longleftarrow g \cdot x = y \cdot h.$$

The action $(G, \mathsf{X}, \mathfrak{M})$ does not depend, up to equivalence of the actions (hence up to conjugacy), on the choice of the basis X.

The action associated to a covering G-bimodule $\mathfrak{M}$ is not faithful in general. The *faithful quotient* of G is the quotient of G by the kernel of the associated action. The action of the faithful quotient on X^* is self-similar and the associated bimodule is called the *faithful quotient* of the bimodule $\mathfrak{M}$.

If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are permutational G-bimodules, then their tensor product $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is the quotient of $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the identifications $x_1 \cdot g \otimes x_2 = x_1 \otimes g \cdot x_2$. It is a G-bimodule with respect to the actions $g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2)$. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are covering bimodules, then $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is also a covering bimodule.

If $\mathfrak{M}$ is a covering bimodule and X is its basis, then the set X^n of words $x_1 x_2 \ldots x_n = x_1 \otimes x_2 \otimes \cdots \otimes x_n$, for $x_i \in \mathsf{X}$, is a basis of the bimodule $\mathfrak{M}^{\otimes n}$. For every $v \in \mathsf{X}^n$ and $g \in G$ there exists then a unique pair $u \in \mathsf{X}^n$ and $h \in G$ such that $g \cdot v = u \cdot h$ in $\mathfrak{M}^{\otimes n}$. The action $g : v \mapsto u$ coincides then with the associated self-similar action $(G, \mathsf{X}, \mathfrak{M})$.

Definition 5. A *virtual endomorphism* of a group G is a homomorphism of groups $\phi : G_1 \longrightarrow G$, where $G_1 < G$ is a subgroup of finite index (called the *domain* of ϕ).

Two virtual endomorphisms ϕ_1, ϕ_2 of G are *conjugate* if there exist $g, h \in G$ such that $h^{-1} \cdot \mathrm{Dom}\,\phi_1 \cdot h = \mathrm{Dom}\,\phi_2$ and $\phi_1(x) = g^{-1}\phi_2(h^{-1}xh)g$ for all $x \in \mathrm{Dom}\,\phi_1$.

If $\mathfrak{M}$ is a covering G-bimodule then, for $x \in \mathfrak{M}$, the *associated virtual endomorphism* ϕ_x is given by the rule

$$g \cdot x = x \cdot \phi_x(g),$$

and is defined on the subgroup of the elements $g \in G$ such that x and $g \cdot x$ belong to one right G-orbit. If the left action of G on the set of right orbits is transitive, then the bimodule $\mathfrak{M}$, and the associated self-similar action are uniquely determined (up

to isomorphism of the bimodules and up to equivalence of self-similar actions) by the associated virtual endomorphism (see [Nek05, Proposition 2.5.8]).

2.2. Contracting groups and their limit spaces.

Definition 6. A self-similar group (G, X) is said to be *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n_0 \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in \mathsf{X}^*$ of length at least n_0.

If the group is contracting, then the smallest set $\mathcal{N}$ satisfying the conditions of the definition is called the *nucleus* of the action.

If a self-similar group is contracting, then every equivalent action is also contracting (though the nucleus may be different). Consequently, the property of being contracting depends only on the associated bimodule, and does not depend on the choice of the basis.

Denote by $\mathsf{X}^{-\omega}$ the set of left-infinite sequences $\ldots x_2x_1$ over the alphabet X with the direct product topology (where X is discrete).

Definition 7. Let (G, X) be a contracting group. We say that two sequences $\ldots x_2x_1, \ldots y_2y_2 \in \mathsf{X}^{-\omega}$ are *asymptotically equivalent* with respect to the action (G, X) if there exists a finite set $N \subset G$ and a sequence $g_k \in N$ such that

$$g_k(x_k \ldots x_2x_1) = y_k \ldots y_2y_1$$

for all k. The quotient of the space $\mathsf{X}^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the action and is denoted $\mathcal{J}_G$.

Proposition 2.2. *The limit space of a contracting self-similar group is a finite dimensional compact metrizable space. The shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$ on $\mathsf{X}^{-\omega}$ induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.*

The dynamical system $(\mathcal{J}_G, \mathsf{s})$ is called the *limit dynamical system* of the contracting group (G, X).

A natural structure of an *orbispace* on $\mathcal{J}_G$ is introduced using the following "covering space" of $\mathcal{J}_G$.

Definition 8. Let (G, X) be a contracting self-similar group. Let $\mathsf{X}^{-\omega} \times G$ be the direct product of the topological space $\mathsf{X}^{-\omega}$ with the discrete group G. We say that $\ldots x_2x_1 \cdot g$ and $\ldots y_2y_1 \cdot h \in \mathsf{X}^{-\omega} \times G$ are *asymptotically equivalent* if there exists a finite set $N \subset G$ and a sequences $g_k \in G$, such that

$$g_k(x_k \ldots x_2x_1) = (y_k \ldots y_2y_1), \quad g_k|_{x_k \ldots x_2x_1} g = h$$

for all k. The quotient of $\mathsf{X}^{-\omega} \times G$ by the asymptotic equivalence relation is called the *limit G-space* and is denoted $\mathcal{X}_G$.

It is easy to see that the natural right action of G on $\mathsf{X}^{-\omega} \times G$ induces an action of G on $\mathcal{X}_G$. The space of orbits $\mathcal{X}_G/G$ of this action is homeomorphic to $\mathcal{J}_G$. The corresponding orbispace is the *limit orbispace* of the contracting group (G, X). For theory of orbispaces, see [BH99, Chapter III.$\mathcal{G}$] and [Nek05, Chapter 4].

For $x \in \mathsf{X}$ and a point $\xi \in \mathcal{X}_G$ represented by a sequence $\ldots x_2x_1 \cdot g$ we denote by $\xi \otimes x$ the point represented by $\ldots x_2x_1g(x) \cdot g|_x$. The map $\xi \mapsto \xi \otimes x$ is continuous.

Definition 9. Let (G, X) be a contracting group. The *tile* $\mathcal{T}$ is the image of the set $\mathsf{X}^{-\omega} \cdot \{1\} \subset \mathsf{X}^{-\omega} \times G$ in the limit G-space $\mathcal{X}_G$.

For $v \in \mathsf{X}^n$ and $g \in G$, the corresponding *tile of nth level* is the set $\mathcal{T} \otimes v \cdot g$, i.e., the image in $\mathcal{X}_G$ of the set of sequences ending by $v \cdot g$.

Proposition 2.3. *Two tiles $\mathcal{T} \otimes v_1 \cdot g_1$ and $\mathcal{T} \otimes v_2 \cdot g_2$ of the nth level intersect if and only if there exists an element h of the nucleus such that $h \cdot v_1 \cdot g_1 = v_2 \cdot g_2$.*

We write $h \cdot v = u \cdot g$, for $v, u \in \mathsf{X}^n$ and $h, g \in G$, if $h(v) = u$ and $g = h|_v$ (which agrees with the definition of the bimodule $\mathfrak{M}^{\otimes n}$). In particular, the equality $h \cdot v_1 \cdot g_1 = v_2 \cdot g_2$ means that $v_2 = h(v_1)$ and $h|_{v_1} g_1 = g_2$.

3. Topological automata

3.1. Definition.

Definition 10. A *topological automaton* is a quadruple $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M}$ and $\mathcal{M}_1$ are topological spaces (or orbispaces), $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a finite covering map and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a continuous map (a morphism of orbispaces).

This definition coincides (in the regular, i.e., non-orbispace case) with the notion of a *topological correspondence* or *topological graph* studied by T. Katsura in [Kat04, Kat06a, Kat06b, Kat08], which might be a better terminology. We use a different term in order to show a strong connection to the theory of self-similar groups and groups generating by automata. We will consider topological automata up to different weak equivalence relations, so that they will be combinatorial rather than rigidly topological objects.

Definition 11. Topological automata $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ and $\mathcal{F}' = (\mathcal{M}', \mathcal{M}'_1, f', \iota')$ are said to be *homotopy equivalent* if there exist homotopy equivalences $\phi_1 : \mathcal{M}'_1 \longrightarrow \mathcal{M}_1$, $\phi : \mathcal{M}' \longrightarrow \mathcal{M}$ and maps $\iota'_1 : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$, $\iota_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$, homotopic to ι' and ι, respectively, such that the diagrams

$$\begin{array}{ccc} \mathcal{M}'_1 & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{M}' & \xrightarrow{\phi} & \mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathcal{M}'_1 & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow{\scriptstyle \iota'_1} & & \downarrow{\scriptstyle \iota_1} \\ \mathcal{M}' & \xrightarrow{\phi} & \mathcal{M} \end{array}$$

are commutative.

We consider topological automata up to homotopy equivalence. We will introduce an even weaker equivalence relation between topological automata later.

Here topological automata are topological analogs of transducers. They should not be confused with analogs of *acceptors* (see, for instance [Bra70, Jea07]).

3.2. Examples of topological automata.

3.2.1. *Automata and Moore diagrams.* Let us recall the definition of *automata* (also known as *transducers*). For more on theory of transducers and groups generated by automata, see [Eil74, GNS00].

Definition 12. An *automaton* over the alphabet X is a triple (Q, π, τ), where Q is a set (of *internal states*), and π and τ are maps

$$\pi : Q \times \mathsf{X} \longrightarrow \mathsf{X}, \quad \tau : Q \times \mathsf{X} \longrightarrow Q,$$

called the *output* and *transition* functions, respectively. The automaton is called *invertible* if for every $q_0 \in Q$ the map $x \mapsto \pi(q_0, x)$ is a permutation. The automaton is *finite* if the set Q is finite.

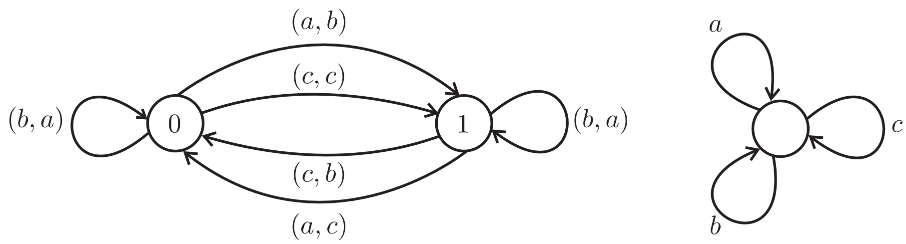

FIGURE 1. Dual Moore diagram

We interpret the automaton (Q, π, τ) as a machine, which being in a state $q \in Q$ and reading on input a letter x prints the letter $\pi(q, x)$ on the output and changes its state to $\tau(q, x)$.

Every invertible automaton can be related to a topological automaton by the notion of a *Moore diagram* (also called a *state diagram*).

Moore diagrams are classical representations of automata. We will use here the *dual Moore diagrams* (i.e., the usual Moore diagrams of the dual automata). It is an oriented graph with the set of vertices X in which for every $x \in \mathsf{X}$ and $q \in Q$ we have an arrow starting in x, ending in $\pi(q, x)$ and labeled by $(q, \tau(q, x))$. See an example of a dual Moore diagram of an automaton on Figure 1.

Dual Moore diagrams are naturally interpreted as topological automata. We take $\mathcal{M}_1$ to be the dual Moore diagram of the automaton (Q, π, τ) as a topological graph (i.e., as a cellular complex). The space $\mathcal{M}$ is a graph with one vertex and $|Q|$ loops labeled by the elements of Q. If an arrow of $\mathcal{M}_1$ is labeled by (q_1, q_2), then it is mapped by $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ to the loop of $\mathcal{M}$ labeled by q_1 and by $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ to the loop labeled by q_2. We get in this way a topological automaton, which is well defined up to a homotopy equivalence. The condition of invertibility of the automaton (Q, τ, π) is equivalent to the condition that f is a covering map.

3.2.2. *Wreath recursions.* More generally, let (G, X) be a finitely generated self-similar group. Choose a generating set S of G. Consider the *dual Moore diagram* of (G, X) with respect to S. It is a directed graph with the set of vertices X in which for every $g \in S$ and $x \in \mathsf{X}$ we have an arrow starting in x, ending in $g(x)$, and labeled by $(g, g|_x)$. This graph describes the wreath recursion of (G, X): the arrows labeled by $(g, \cdot)$ describe the action of the generator g on X, and the second coordinates of the labels show the corresponding sections $g|_x$.

The dual Moore diagram of a self-similar group is a topological automaton. The graph $\mathcal{M}$, as in the previous example, has one vertex and oriented loops labeled by the elements of S. Let $\mathcal{M}_1$ be the dual Moore diagram of (G, X). The first coordinates of the labels show the values of the covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$; the second coordinates show the values of the map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$: the arrow labeled by (g, h) is mapped by ι to the path in $\mathcal{M}$ such that the product of the labels along the path (taking into account the orientation) is equal to h. Note that the obtained topological automaton is not uniquely defined even up to a homotopy equivalence, since elements h of G may be represented in different ways as products of the elements of S.

3.2.3. *Partial self-coverings.* If ι is an embedding, then the topological automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ is a *partial self-covering* of $\mathcal{M}$. Partial self-coverings are studied

in [Nek05]. See also [Nek08b], where the category of partial self-coverings is defined. Theory of topological automata and their iterated monodromy groups is not much different from the theory of partial self-coverings. The main reason to introduce the general notion (except for the pure sake of generality) is that topological automata are less rigid objects, and are easier to construct, and hence to use them as models of more complicated partial self-coverings and their Julia sets.

3.2.4. *Post-critically finite rational functions.* A rational function $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is said to be *post-critically finite* if the orbit of every critical point of f under iterations of f is finite. Denote by P the *post-critical set* of f, i.e., the union of the orbits of critical values of f. Then $f : \widehat{\mathbb{C}} \setminus f^{-1}(P) \longrightarrow \widehat{\mathbb{C}} \setminus P$ is a partial self-covering (since $\widehat{\mathbb{C}} \setminus f^{-1}(P) \subseteq \widehat{\mathbb{C}} \setminus P$). Hence post-critically finite rational functions are examples of topological automata.

3.2.5. *Post-critically finite correspondences.* Let $R \subset \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ be a correspondence, i.e., an algebraic curve in $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. Denote by p_1 and p_2 projections of R onto the first and the second coordinates of the correspondence. We assume that p_1 and p_2 are branched coverings and interpret R as a multivalent function

$$p_2(z) \mapsto p_1(z).$$

Suppose that R is *post-critically finite*, i.e., there exists a finite set $P \subset R$ such that $p_1 : R \setminus P \longrightarrow \widehat{\mathbb{C}} \setminus p_1(P)$ is a covering and $p_1(P) \subseteq p_2(P)$.

We have $p_2(R \setminus P) \subseteq p_1(R \setminus P)$, hence the quadruple $(p_1(R \setminus P), R \setminus P, p_1, p_2)$ is a topological automaton.

As a simple example, consider the correspondence

$$z^q \mapsto z^p$$

for natural numbers p and q, i.e., the multivalent function $z^{p/q}$. Its post-critical set is $\{0, \infty\}$.

Another famous example is the correspondence associated with the *arithmetic-geometric mean*, studied by Gauss. An extensive account on the history of arithmetic-geometric mean is given in [Cox84].

Lagrange in 1784 and independently Gauss in 1790 have shown that if a_0 and b_0 are positive real numbers, then the sequences

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \quad b_n = \sqrt{a_{n-1} b_{n-1}}$$

converge to a common value $M(a_0, b_0)$, called the arithmetic-geometric mean.

In the complex case one has to choose one of two signs of the square root. We get the correspondence

$$[z_1 : z_2] \mapsto [(z_1 + z_2)/2 : \sqrt{z_1 z_2}]\,,$$

on the projective line $\widehat{\mathbb{C}}$. It is written in the affine coordinates as

$$w \mapsto \frac{1 + w}{2\sqrt{w}}.$$

In our terms, the correspondence is given by the pair of maps

$$f(w) = \frac{(1 + w)^2}{4w}, \quad \iota(w) = w^2,$$

so that it is the curve $\left\{\left(\frac{(1+w)^2}{4w}, w^2\right) : w \in \widehat{\mathbb{C}}\right\}$. Denote by P the set of the points $(\infty, 0), (1, 1), (0, 1)$ and (∞, ∞), which are the points of R parametrized by $w = 0, 1, -1$ and ∞, respectively. We have

$$f(P) = \{\infty, 1, 0\} = \iota(P),$$

and the maps $f, \iota : R \setminus P \longrightarrow \widehat{\mathbb{C}} \setminus \pi_1(P)$ are coverings.

See [Bul91], where the arithmetic-geometric mean is studied as an example of a post-critically finite correspondence. For more on dynamics of correspondences see the papers [Bul88, Bul92, BP94].

3.2.6. *Bi-reversible automata.* In the last example both maps f and ι were coverings. This situation for the dual Moore diagrams of automata has a special name.

Definition 13. Let $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ be the dual Moore diagram of a finite invertible automaton. The automaton is said to be *bi-reversible* if the map ι is a covering.

An example of a bi-reversible automaton (of its dual Moore diagram) is shown on Figure 1. It corresponds to one of two automata, which appeared in the paper [Ale83]. It was proved in [VV07] that the self-similar group generated by this automaton is free. For more on bi-reversible automata see [MNS00, GM05, VV07] and Section 1.10 of [Nek05].

3.2.7. *Commensurizers of tree lattices.* If G is a group and $H < G$ is a subgroup, then the *commensurizer* of H in G is the group of the elements $g \in G$ such that $H \cap g^{-1}Hg$ has finite index in H and $g^{-1}Hg$.

As a generalization of bi-reversible automata, consider the topological automata $(\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M}$ is a bouquet of k circles, and $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ are coverings. It is shown in Proposition 2.2 of [LMZ94] that the topological automata $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ of this form describe the elements of the commensurizer of the co-compact lattice $\pi_1(\mathcal{M})$ in the automorphism group of the universal covering T of $\mathcal{M}$.

More precisely, if g is an element of the commensurizer of $\pi_1(\mathcal{M}, t)$ in the automorphism group of T, then there exists a finite index subgroup $H < \pi_1(\mathcal{M})$ such that $f(e) = \iota(g(e))$ for every edge e of T. It follows that g is uniquely determined by the image $g(t_0)$ of a vertex t_0 of T and by the automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ (which is called a *periodic recoloring* in [LMZ94]).

For more on lattices in the automorphism groups of trees, see [BL01, GM05].

3.2.8. *Thurston maps.* A *Thurston map* is a post-critically finite orientation preserving branched self-covering $f : S^2 \longrightarrow S^2$ of the sphere. It can be interpreted as a topological automaton in the same way as in the case of post-critically finite rational functions.

Thurston's theorem (see [DH93]) gives a criterion when a Thurston map is equivalent to a post-critically finite rational function. Definition of homotopy equivalence (Definition 11) is a generalization of the equivalence relation introduced in Thurston's theorem.

In many cases it is more convenient not to remove all post-critical points from the sphere S^2, but rather to introduce an orbifold structure on S^2 minus some post-critical points. The corresponding orbifold construction, also due to Thurston, is defined as follows.

Let C_f be the set of critical points of a Thurston map $f : S^2 \longrightarrow S^2$ and let $P_f = \bigcup_{n\ge 1} f^n(C_f)$ be the post-critical set. Let $P' \subset P_f$ be the union of all cycles of f intersecting C_f (they are *superattracting* if f is a rational function).

The underlying space of the orbifold $\mathcal{M}$ will be $S^2 \setminus P'$. The points $P_f \setminus P'$ will be its singular points.

Denote by $\nu(x)$ for $x \in S^2 \setminus P'$ the least common multiple of the local degrees of f^m at z, for all z such that $f^m(z) = x$. The number $\nu(x)$ is finite for all $x \in S^2 \setminus P'$ and greater than 1 if and only if $x \in P_f$.

Then for any $x \in S^2$ the number $\nu(f(x))$ is divisible by $\deg_x(f) \cdot \nu(x)$, where $\deg_x(f)$ denotes the local degree of f at x.

Let $\mathcal{M}$ be the orbispace with the underlying space $S^2 \setminus P'$ for which a point $x \in \mathcal{M}$ is uniformized in the atlas of the orbispace by the cyclic group of order $\nu(x)$ acting by rotations of a disc.

Similarly, let $\mathcal{M}_1$ be the orbifold defined by the weights $\nu_0(x) = \frac{\nu(f(x))}{\deg_x(f)}$ instead of $\nu(x)$. The set of singular points of $\mathcal{M}_1$ is contained in $f^{-1}(P_f)$. The underlying space of $\mathcal{M}_1$ is $S^2 \setminus f^{-1}(P')$.

We have $\nu(z)|\nu_0(z)$, hence the orbispace $\mathcal{M}_{\nu_0}$ is an open sub-orbispace of $\mathcal{M}_\nu$, where the embedding is identical on the underlying spaces (see a definition of embedding of orbifolds in [Nek05, Section 4.3]).

On the other hand, the map $f : \mathcal{M}_{\nu_0} \longrightarrow \mathcal{M}_\nu$ is a covering of the orbispaces, since $\deg_x(f) = \frac{\nu_0(x)}{\nu(f(x))}$.

In this way we get an orbifold topological automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$, where ι is the identical embedding of the orbispaces.

3.2.9. *Subdivision rules.* Finite subdivision rules are convenient combinatorial descriptions of Thurston maps, see [CFP01, CFP07].

See [CFP01] for a precise definition of subdivision rules. In our terminology, subdivision rules correspond to topological automata $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are two-dimensional CW complexes (or complexes of groups), $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a cellular covering map and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a homeomorphism such that $\iota(\mathcal{M}_1)$ is a subdivision of $\mathcal{M}$.

The cells of $\mathcal{M}$ are called *types.* Description of the covering f amounts to prescribing types (i.e., images under f) to the cells of $\mathcal{M}_1$. The subdivision rule specifies then how the cells of $\mathcal{M}$ are subdivided into the images of the cells of $\mathcal{M}_1$ under ι, i.e., specifies the subdivision and labels the cells according to their types. One also has to label the edges and vertices appropriately, so that one gets uniquely defined maps f and ι.

4. Iterated monodromy groups

4.1. **Iteration of topological automata.** Every topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ can be *iterated* in the following way. Denote $\mathcal{M}_0 = \mathcal{M}$, $f_0 = f$, and $\iota_0 = \iota$. Define inductively the covering $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ as the pullback of the covering $f_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}_{n-1}$ by the map $\iota_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}_{n-1}$, and the map $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ as the morphism closing the pullback diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_n} & \mathcal{M}_n \\ \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle f_{n-1}} \\ \mathcal{M}_n & \xrightarrow{\iota_{n-1}} & \mathcal{M}_{n-1}. \end{array}$$

Then the nth iteration $\mathcal{F}^n$ of the topological automaton $\mathcal{F}$ is the pair of maps

$$f^n = f_0 \circ f_1 \circ \cdots \circ f_{n-1}, \quad \iota^n = \iota_0 \circ \iota_1 \circ \cdots \circ \iota_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}.$$

In the case when $\mathcal{M}$ and $\mathcal{M}_1$ are regular (i.e., are usual topological spaces), the pullback $\mathcal{M}_2$ can be defined as the subspace

$$\{(x, y) \in \mathcal{M}_1^2 \ : \ f(y) = \iota(x)\},$$

so that the map $\iota_1 : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ and the covering $f_1 : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ are given by $\iota_1(x, y) = y$ and $f_1(x, y) = x$.

We get hence by induction the following description of the iteration.

Proposition 4.1. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton such that $\mathcal{M}$ (and hence $\mathcal{M}_1$) are regular. Then the space $\mathcal{M}_n$ is homeomorphic to the subspace*

$$\{(x_1, x_2, \ldots, x_n) \in \mathcal{M}_1^n \ : \ f(x_{i+1}) = \iota(x_i), i = 1, \ldots, n-1\},$$

and the maps $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ and $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ are given by

$$f_n(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_n),$$

and

$$\iota_n(x_1, x_2, \ldots, x_{n+1}) = (x_2, x_3, \ldots, x_{n+1}).$$

In particular, the topological automaton $\mathcal{F}^n$ is defined by the maps

$$f^n(x_1, x_2, \ldots, x_n) = f(x_1),$$

and

$$\iota^n(x_1, x_2, \ldots, x_n) = \iota(x_n).$$

Example 1. If $\mathcal{F}$ is the automaton defined by a covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ of a topological space by its open subset, then $\mathcal{M}_n$ is the domain of the nth iterate f^n of f and the automaton $\mathcal{F}^n$ is defined by the partial self-covering $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$.

Example 2. If the topological automaton is the dual Moore diagram of an automaton, then the topological automaton $\mathcal{F}^n = (\mathcal{M}, \mathcal{M}_n, f^n, \iota^n)$ is the dual Moore diagram of the same automaton over the alphabet X^n. Analogous statement holds for topological automata describing wreath recursions on groups.

4.2. **Three inverse limits of a topological automaton.** Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton. Iterations of $\mathcal{F}$ produce the following infinite commutative diagram of topological spaces. We consider $\mathcal{M}_n$ as regular topological spaces even if $\mathcal{M}$ is an orbispace, i.e., consider the underlying spaces only.

$$\begin{array}{ccccccc}
\ddots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} \\
\ldots & \xrightarrow{\iota_4} & \mathcal{M}_4 & \xrightarrow{\iota_3} & \mathcal{M}_3 & \xrightarrow{\iota_2} & \mathcal{M}_2 \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
\ldots & \xrightarrow{\iota_3} & \mathcal{M}_3 & \xrightarrow{\iota_2} & \mathcal{M}_2 & \xrightarrow{\iota_1} & \mathcal{M}_1 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
\ldots & \xrightarrow{\iota_2} & \mathcal{M}_2 & \xrightarrow{\iota_1} & \mathcal{M}_1 & \xrightarrow{\iota} & \mathcal{M}
\end{array}$$

Denote by $\lim_f \mathcal{F}$ the inverse limit of the columns of this diagram. The limit obviously does not depend on the choice of the column and the maps ι_n between the columns induce a continuous map $\iota_\infty : \lim_f \mathcal{F} \longrightarrow \lim_f \mathcal{F}$. Similarly, denote by

$\lim_\iota \mathcal{F}$ the inverse limit of the rows. The maps f_n induce then a continuous map $f_\infty : \lim_\iota \mathcal{F} \longrightarrow \lim_\iota \mathcal{F}$, which is a covering in the regular case.

We may also consider the inverse limit of the whole diagram, which we will denote $\lim_{f,\iota} \mathcal{F}$. The "diagonal" identical map between the corners $\mathcal{M}_n$ of the commutative squares

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_n} & \mathcal{M}_n \\ \downarrow f_n & \swarrow & \downarrow f_{n-1} \\ \mathcal{M}_n & \xrightarrow{\iota_{n-1}} & \mathcal{M}_{n-1} \end{array}$$

induces a homeomorphism Δ of $\lim_{f,\iota} \mathcal{F}$. The following is straightforward.

Proposition 4.2. *The space $\lim_{f,\iota} \mathcal{F}$ is homeomorphic to the inverse limit of the sequence*

$$\ldots \xleftarrow{f_\infty} \lim_\iota \mathcal{F} \xleftarrow{f_\infty} \lim_\iota \mathcal{F} \xleftarrow{f_\infty} \lim_\iota \mathcal{F},$$

and to the inverse limit of the sequence

$$\ldots \xleftarrow{\iota_\infty} \lim_f \mathcal{F} \xleftarrow{\iota_\infty} \lim_f \mathcal{F} \xleftarrow{\iota_\infty} \lim_f \mathcal{F}.$$

The homeomorphism Δ is induced by the action of f_∞ on the first inverse limit and the homeomorphism Δ^{-1} is induced by the action of ι_∞ on the second inverse limit.

Definition 14. Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a regular topological automaton. A *forward $\mathcal{F}$-orbit* is a sequence $x_1, x_2, \ldots,$ of points of $\mathcal{M}_1$ such that

$$f(x_n) = \iota(x_{n+1})$$

for all $n = 1, 2, \ldots$.

A *backward $\mathcal{F}$-orbit* is a sequence $x_1, x_2, \ldots,$ of points of $\mathcal{M}_1$ such that

$$f(x_{n+1}) = \iota(x_n)$$

for all $n = 1, 2, \ldots$.

A *bilateral $\mathcal{F}$-orbit* is a sequence $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ such that

$$f(x_n) = \iota(x_{n+1})$$

for all $n \in \mathbb{Z}$.

The choice of the names "backward" and "forward" is almost arbitrary. We use the choice given in the definition, since iteration of partial self-coverings is our main motivation. It is, however, also natural to use the opposite terminology, like it is done in [Kat04, Kat06a], especially in the setting of automata theory, groupoids or operator algebras.

The spaces of forward, backward and bilateral $\mathcal{F}$-orbits is endowed with the topology of a subset of the corresponding direct powers of $\mathcal{M}_1$.

The following description of the inverse limits is a direct corollary of Proposition 4.1.

Proposition 4.3. *The spaces $\lim_\iota \mathcal{F}$, $\lim_f \mathcal{F}$ and $\lim_{f,\iota} \mathcal{F}$ are homeomorphic to the spaces of forward, backward and bilateral $\mathcal{F}$-orbits, respectively. The maps f_∞, ι_∞ and Δ are induced by the shifts on the corresponding spaces of orbits.*

Example 3. If $\mathcal{M}$ is a topological space and ι is an embedding (i.e., if $\mathcal{F}$ is a partial self-covering), then $\lim_\iota \mathcal{F}$ is the intersection of the domains $\mathcal{M}_n$ of f^n.

Example 4. Let $f \in \mathbb{C}(z)$ be a hyperbolic rational function. Let $U \subset \widehat{\mathbb{C}}$ be a closed set such that U does not intersect the union of the attracting cycles of f, U contains the Julia set of f, and $f^{-1}(U) \subseteq U$. Let $\mathcal{F} = (U, f^{-1}(U), f, id)$ be the corresponding topological automaton. Then $\lim_{id} \mathcal{F}$ is the Julia set of f, and f_∞ is the restriction of f onto its Julia set.

The space of backward orbits $\lim_f \mathcal{F}$ for rational functions was studied (in greater generality) in [LM97, KL05].

4.3. **Definition of the iterated monodromy group.** The definition of the iterated monodromy group of a topological automaton almost coincides with the definition of the iterated monodromy group of a partial self-covering (especially in its orbispace version). Here we give a short overview of the definitions for regular (non-orbispace) case. For more details and for the definition in the case of orbispace topological automata, see [Nek05] (the map ι is considered in [Nek05] to be an embedding of orbispaces, but this fact is never used).

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton, and suppose that $\mathcal{M}$ is path connected and locally path connected.

Choose a basepoint $t \in \mathcal{M}$, and consider the sequence of the coverings

$$\mathcal{M} \xleftarrow{f} \mathcal{M}_1 \xleftarrow{f_1} \mathcal{M}_2 \xleftarrow{f_2} \cdots,$$

and denote $f^n = f \circ f_1 \circ \cdots \circ f_{n-1}$, and $f^{-n} = (f^n)^{-1}$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts on each of the sets $f^{-n}(t) \subset \mathcal{M}_n$ by the monodromy action: the image of a point $z \in f^{-n}(t)$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ is the endpoint of the unique lift of γ by f^n that starts at z.

The union $T = \bigcup_{n \geq 0} f^{-n}(t)$ is called the *preimage tree* of the point t. We define vertex adjacency in T in the natural way, so that a point $z \in f^{-n}(t)$ is connected by an edge to the point $f_{n-1}(z) \in f^{-(n-1)}(t)$. It is easy to see that the action of the fundamental group on the sets $f^{-n}(t)$ is an action by automorphisms of the preimage tree. This action is called the *iterated monodromy action*.

Definition 15. The *iterated monodromy group* of a topological automaton $\mathcal{F}$ is the quotient of the fundamental group of $\mathcal{M}$ by the kernel of its action on the tree of preimages.

4.4. **Coding tree.** Exactly as in the case of partial self-coverings, the iterated monodromy group of a topological automaton can be computed using a natural self-similarity structure on it. We repeat here the constructions of [Nek05, Sections 5.1–2] simplified to the case of regular topological automata.

For a covering $P : \mathcal{X}_1 \longrightarrow \mathcal{X}$, a path γ in $\mathcal{X}$ and a preimage $z \in \mathcal{X}_1$ of the beginning of γ, we denote by $P^{-1}(\gamma)_z$ the lift of γ by P starting at z.

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton with a path connected and locally path connected base space $\mathcal{M}$. Let X be an alphabet of size equal to the degree of the covering f. Choose a basepoint $t \in \mathcal{M}$ and a bijection $\Lambda_1 : \mathsf{X} \longrightarrow f^{-1}(t)$. Also choose for every $x \in \mathsf{X}$ a path ℓ_x in $\mathcal{M}$ starting at t and ending in $\iota(\Lambda_1(x))$.

Define now inductively the points $\Lambda_1(v) \in \mathcal{M}_1$, curves ℓ_v^1 in $\mathcal{M}_1$ for $v \in \mathsf{X}^* \setminus (\mathsf{X}^1 \cup \mathsf{X}^0)$, and ℓ_v in $\mathcal{M}$ for $v \in \mathsf{X}^*$, by the rules

$$\ell^1_{xvy} = f^{-1}(\ell_{xv})_{\Lambda_1(vy)}, \quad \ell_v = \iota(\ell^1_v), \quad \Lambda_1(v) \text{ is the end of } \ell^1_v. \tag{3}$$

In other words, we lift the curves of $\mathcal{M}$ by f to curves in $\mathcal{M}_1$ and then push them back into $\mathcal{M}$ by ι. In this way we get a tree of curves ℓ_v in $\mathcal{M}$ with the root in t and d trees of curves ℓ_v^1 with the roots in $f^{-1}(t)$.

The curve ℓ_{xv}^1 connects $\Lambda_1(v)$ with $\Lambda_1(xv)$; the curve ℓ_{xv} connects the point $\iota(\Lambda_1(v))$ with the point $\iota(\Lambda_1(xv))$. It follows from the definition that

$$f(\Lambda_1(x_1 \ldots x_k)) = \iota(\Lambda_1(x_1 \ldots x_{k-1})),$$

since the curve $\ell_{x_1 \ldots x_k}^1$ ends in $\Lambda_1(x_1 \ldots x_k)$, while its f-image

$$f(\ell_{x_1 \ldots x_k}^1) = \ell_{x_1 \ldots x_{k-1}} = \iota(\ell_{x_1 \ldots x_{k-1}}^1)$$

ends in $\iota(\Lambda(x_1 \ldots x_{k-1}))$. Consequently, by Proposition 4.1, the sequence

$$(\Lambda_1(x_1), \Lambda_1(x_1x_2), \ldots, \Lambda_1(x_1x_2 \ldots x_{n-1}), \Lambda_1(x_1x_2 \ldots x_n))$$

defines a point of $\mathcal{M}_n$, which we will denote by $\Lambda(x_1x_2 \ldots x_n)$.

Proposition 4.4. *The map $\Lambda : \mathsf{X}^* \longrightarrow T$ is an isomorphism of the tree of words with the preimage tree.*

We call the isomorphism Λ the *coding* of the preimage tree, defined by the connecting paths ℓ_x.

Proof. A direct corollary of the construction and Proposition 4.1. □

4.5. **Computation of the iterated monodromy group.**

Theorem 4.5. *Let $\Lambda : \mathsf{X}^* \longrightarrow T$ be the coding defined by a collection of paths ℓ_x connecting the basepoint t to $\iota(\Lambda(x))$. Then the action of $\pi_1(\mathcal{M}, t)$ on X^*, obtained by conjugation of the iterated monodromy action by the isomorphism Λ, is defined by the following recurrent rule:*

$$\gamma(xv) = y \left(\ell_y^{-1} \iota(f^{-1}(\gamma)_{\Lambda(x)}) \ell_x\right)(v),$$

where $y = \gamma(x)$ is the end of $f^{-1}(\gamma)_{\Lambda(x)}$.

Remark. Here and throughout the paper we multiply paths as functions: in a product $\gamma_1 \cdot \gamma_2$ the path γ_2 is passed before γ_1.

The proof of the above theorem is the same as in the case of partial self-coverings, see [Nek05, Proposition 5.2.2].

Definition 16. The self-similar action of the iterated monodromy group on X^* described in Theorem 4.5 is called the *standard action* (*defined by the bijection* $\Lambda_1 : \mathsf{X} \longrightarrow f^{-1}(t)$ *and connecting paths* ℓ_x).

Proposition 4.6. *The standard action of the iterated monodromy group does not depend on the choice of the bijection Λ_1 and the connecting paths ℓ_x, up to equivalence of self-similar groups. Any self-similar action equivalent to a standard action is a standard action.*

In other words, the iterated monodromy group $\mathrm{IMG}(\mathcal{F})$ has a natural well defined self-similarity structure.

Proof. A change of the bijection is equivalent to post-conjugation of the wreath recursion by an element of $\mathfrak{S}(\mathsf{X})$. Changing the set of connecting paths $(\ell_x)_{x \in \mathsf{X}}$ to a set $(\ell'_x)_{x \in \mathsf{X}}$ (for a fixed bijection Λ_1) corresponds, by Theorem 4.5, to post-conjugating the wreath recursion by the element $(\ell_x^{-1}\ell'_x)_{x \in \mathsf{X}} \in (\pi_1(\mathcal{M}, t)^{\mathsf{X}}$. □

Standard actions of the iterated monodromy groups can be also defined using the associated virtual endomorphisms.

Let $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ be, as before, a topological automaton with a path connected and locally path connected space $\mathcal{M}$. We will assume now that $\mathcal{M}_1$ is also path connected. Fix some basepoint $t \in \mathcal{M}$ and a point $t_1 \in f^{-1}(t)$. Choose a path ℓ in $\mathcal{M}$ from t to $\iota(t_1)$. Let G_1 be the subgroup of $\pi_1(\mathcal{M}, t)$ of loops γ such that the lift $f^{-1}(\gamma)_{t_1}$ is also a loop. The subgroup G_1 is of index d in $\pi_1(\mathcal{M}, t)$ (and is isomorphic to $\pi_1(\mathcal{M}_1)$).

Definition 17. The *virtual endomorphism* of $\pi_1(\mathcal{M}, t)$, *associated* with the topological automaton is the homomorphism

$$\phi : G_1 \longrightarrow \pi_1(\mathcal{M}, t) : \gamma \mapsto \ell^{-1}\iota(f^{-1}(\gamma)_{t_1})\ell.$$

It is easy to check that the associated endomorphism does not depend, up to conjugacy of virtual endomorphisms, on the choice of the preimage t_1 and of the connecting path ℓ. Moreover, it does not depend on the choice of the basepoint t, if we identify the fundamental groups with different basepoints in the standard way, using connecting paths.

Similarly to the case of partial self-coverings (see [Nek05, Proposition 5.1.2]), one can show that the standard action of the iterated monodromy group of a topological automaton is equivalent to the self-similar action defined by the associated virtual endomorphism.

4.6. **Combinatorial equivalence.**

Definition 18. We say that two topological automata with path-connected base (orbi)spaces are *combinatorially equivalent* if their iterated monodromy groups are equivalent as self-similar groups.

Here we consider faithful iterated monodromy groups, and not just the self-similarity on the fundamental group.

Proposition 4.7. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ and $\mathcal{F}' = (\mathcal{M}', \mathcal{M}'_1, f', \iota')$ be topological automata with path-connected base spaces $\mathcal{M}$ and $\mathcal{M}'$. Let $\phi : \mathcal{M}' \longrightarrow \mathcal{M}$ be a continuous map and suppose that the covering $f' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ is the pullback of f by ϕ. Let then $\phi_1 : \mathcal{M}'_1 \longrightarrow \mathcal{M}_1$ be the map making the diagram*

$$\begin{array}{ccc} \mathcal{M}'_1 & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \Big\downarrow f' & & \Big\downarrow f \\ \mathcal{M}' & \xrightarrow{\phi} & \mathcal{M} \end{array}$$

commutative. Suppose that the diagram

$$\begin{array}{ccc} \pi_1(\mathcal{M}'_1) & \xrightarrow{(\phi_1)_*} & \pi_1(\mathcal{M}_1) \\ \Big\downarrow \iota'_* & & \Big\downarrow \iota_* \\ \pi_1(\mathcal{M}') & \xrightarrow{\phi_*} & \pi_1(\mathcal{M}) \end{array} \tag{4}$$

is also commutative up to an inner automorphism of $\pi_1(\mathcal{M})$, and the map ϕ_* is an epimorphism. Then the topological automata $\mathcal{F}$ and $\mathcal{F}'$ are combinatorially equivalent.

In particular, homotopically equivalent topological automata are combinatorially equivalent.

Proof. Let $t' \in \mathcal{M}'$ and $t \in \mathcal{M}$ be such that $t = \phi(t')$. Choose a bijection $\Lambda'_1 : \mathsf{X} \longrightarrow (f')^{-1}(t')$ and a collection of connecting paths ℓ_x, defining the standard action of the iterated monodromy group of $\mathcal{F}'$. Then $\phi_1 \circ \Lambda'_1$ and $\phi(\ell_x)$ is a bijection and a collection of connecting paths, defining some standard action of $\mathrm{IMG}(\mathcal{F})$. Both standard actions are unique up to an equivalence of self-similar actions. Let $\psi' : \pi_1(\mathcal{M}') \longrightarrow \pi_1(\mathcal{M}')^{\mathsf{X}} \rtimes \mathfrak{S}(\mathsf{X})$ and $\psi : \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{M})^{\mathsf{X}} \rtimes \mathfrak{S}(\mathsf{X})$ be the associated wreath recursions.

It follows then from commutativity of the diagram (4) and Theorem 4.5 that if $\phi_*(g') = g$, then $\psi(g)$ is obtained from $\psi'(g')$ by applying ϕ_* and a fixed inner automorphism of $\pi_1(\mathcal{M})$ to every coordinate of $\pi_1(\mathcal{M}')^{\mathsf{X}}$. But this implies that the iterated monodromy groups of $\mathcal{F}'$ and $\mathcal{F}$ are equivalent. □

4.6.1. *Moore diagrams of the standard action.* The process of finding the standard self-similarity on the iterated monodromy group $\mathrm{IMG}(\mathcal{F})$ is naturally interpreted, using Proposition 4.7, as passing to a topological automaton that is a Moore diagram of a self-similar group and is combinatorially equivalent to $\mathcal{F}$.

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton such that $\mathcal{M}$ is path connected and locally simply connected. Let $S = \{\gamma_i\}_{i \in I}$ be a generating set of the fundamental group $\pi_1(\mathcal{M}, t)$. Let Γ be a rose of loops g_i with a basepoint t_0, for $i \in I$ and let $\phi : \Gamma \longrightarrow \mathcal{M}$ be a map such that $\phi(g_i) = \gamma_i$, $\phi(t_0) = t$. Then the map $\phi : \Gamma \longrightarrow \mathcal{M}$ induces a surjective map of the fundamental groups.

Let the covering $f' : \Gamma_1 \longrightarrow \Gamma$ and the map $\phi_1 : \Gamma_1 \longrightarrow \mathcal{M}_1$ be obtained by taking pullback of the covering f by the map ϕ. If we find a map $\iota' : \Gamma_1 \longrightarrow \Gamma$ making the diagram

$$\begin{array}{ccc} \pi_1(\Gamma_1) & \xrightarrow{\phi_{1*}} & \pi_1(\mathcal{M}_1) \\ \Big\downarrow{\scriptstyle \iota'_*} & & \Big\downarrow{\scriptstyle \iota_*} \\ \pi_1(\Gamma) & \xrightarrow{\phi_*} & \pi_1(\mathcal{M}) \end{array} \tag{5}$$

commutative, then the one-dimensional topological automaton $(\Gamma, \Gamma_1, f', \iota')$ will be combinatorially equivalent to $\mathcal{F}$, by Propositions 4.7.

If ι' is such that $\iota'((f')^{-1}(t_0)) = \{t_0\}$, then the automaton $(\Gamma, \Gamma_1, f', \iota')$ is the dual Moore diagram of a wreath recursion that defines a standard action of $\mathrm{IMG}(\mathcal{F})$, by Propositions 4.7 and 4.6. Conversely, any dual Moore diagram associated with the wreath recursion of a standard action of $\mathrm{IMG}(\mathcal{F})$ can be obtained in this way. If the standard action is defined by connecting paths ℓ_x, then for a lift $h_z \subset \Gamma_1$ of a loop g_i of Γ we define $\iota'(h_z)$ to be a loop g such that $\phi(g) = \ell_x^{-1} \iota(\phi_1(h_z)) \ell_y$, where y is the beginning and x is the end of $\phi_1(h_z)$. It is easy to check that so defined map ι' makes the diagram (5) commutative up to an inner automorphism of $\pi_1(\mathcal{M})$.

Consequently, Proposition 4.7 is a complete description of combinatorial equivalence. Two automata are combinatorially equivalent if and only if there exists a third automaton, combinatorial equivalence of which to the first two can be established using Proposition 4.7.

5. Contracting automata

5.1. **Self-similar G-spaces.** Let us redefine the notion of a topological automaton in terms of actions of groups on topological spaces. This approach will help us to

use the techniques of self-similar groups, and will include orbispace automata into our consideration without heavy use of the theory of orbispaces.

Let $\mathfrak{M}$ be a covering bimodule over a group G (see Definition 4) and let $\mathcal{X}$ be a topological space with a right action of G by homeomorphisms. Then the tensor product $\mathcal{X}\otimes\mathfrak{M}$ is defined as the quotient of the direct product $\mathcal{X}\times\mathfrak{M}$ of topological spaces (where $\mathfrak{M}$ is discrete) by the identifications

$$\xi \cdot g \otimes x = \xi \otimes g \cdot x.$$

The space $\mathcal{X}\times\mathfrak{M}$ is a right G-space with respect to the action

$$(\xi \otimes x) \cdot g = \xi \otimes (x \cdot g).$$

Definition 19. A right G-space $\mathcal{X}$ is said to be $\mathfrak{M}$-*invariant* (or *self-similar*) if the right G-spaces $\mathcal{X}$ and $\mathcal{X}\otimes\mathfrak{M}$ are conjugate, i.e., if there exists a G-equivariant homeomorphism $I : \mathcal{X}\otimes\mathfrak{M} \longrightarrow \mathcal{X}$. It is called $\mathfrak{M}$-*semi-invariant* if there exists a G-equivariant continuous map $I : \mathcal{X}\otimes\mathfrak{M} \longrightarrow \mathcal{X}$

An example of a self-similar G-space for a contracting group G is the limit G-space $\mathcal{X}_G$, where the conjugacy I maps $\xi\otimes x$, for $\xi \in \mathcal{X}_G$ and $x \in \mathsf{X}$, to the point of $\mathcal{X}_G$ represented by $\ldots x_2x_1g(x)\cdot g|_x$, if ξ is represented by $\ldots x_2x_1 \cdot g$ (see [Nek05, Section 3.4] and Subsection 2.2 of our paper).

Lemma 5.1. *Let $\mathcal{X}_1, \mathcal{X}_2$ be locally compact, Hausdorff, proper, and co-compact right G-spaces. Then every G-equivariant map $\Phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ is proper, i.e., $\Phi^{-1}(C)$ is compact for every compact $C \subset \mathcal{X}_2$.*

Recall that an action of G on $\mathcal{X}$ is said to be proper if for every compact subset $C \subset \mathcal{X}$ the set of elements $g \in G$ such that $C \cdot g \cap C \neq \emptyset$ is finite. It is called co-compact if there exists a compact set K intersecting every G-orbit.

Proof. Let $K \subset \mathcal{X}_1$ be a compact set such that $\mathcal{X}_1 = \bigcup_{g\in G} K \cdot g$. Let $C \subset \mathcal{X}_2$ be any compact set. The set $A = \{g \in G \ : \ \Phi(K)\cdot g \cap C \neq \emptyset\}$ is finite by compactness of $\Phi(K) \cup C$ and properness of the action of G on $\mathcal{X}_2$. Then

$$\Phi^{-1}(C) \subseteq \bigcup_{g\in A} K \cdot g,$$

hence $\Phi^{-1}(C)$ is compact. □

Lemma 5.2. *Let $\mathcal{X}$ be a locally compact Hausdorff right G-space and let $\mathfrak{M}$ be a covering G-bimodule.*

If the action of G on $\mathcal{X}$ is proper and co-compact, then the action of G on $\mathcal{X}\otimes\mathfrak{M}$ is also proper and co-compact.

Proof. Let $K \subset \mathcal{X}$ be an open set with compact closure such that $\mathcal{X} = \bigcup_{g\in G} K \cdot g$. Then every point of $\mathcal{X}\otimes\mathfrak{M}$ can be written in the form $\xi \otimes x$ for $\xi \in K$ and $x \in \mathfrak{M}$. Let X be a basis of $\mathfrak{M}$. Then $\mathcal{X}\otimes\mathfrak{M} = \bigcup_{g\in G}(K \otimes \mathsf{X}) \cdot g$. The set $K \otimes \mathsf{X}$ is compact, hence the action of G on $\mathcal{X}\otimes\mathfrak{M}$ is co-compact.

For every $x \in \mathfrak{M}$ the set $K \otimes x \subset \mathcal{X}\otimes\mathfrak{M}$ is open, since its preimage

$$\bigcup_{g\in G} (K \cdot g, g^{-1} \cdot x)$$

in $\mathcal{X}\times\mathfrak{M}$ is open.

Let $C \subset \mathcal{X}\otimes\mathfrak{M}$ be a compact set. There exists a finite set $M \subset \mathfrak{M}$ such that $C \subset \bigcup_{x\in M} K \otimes x$. Suppose that $\xi_1, \xi_2 \in C$ and $g \in G$ are such that $\xi_1 \cdot g = \xi_2$.

Then there exist $\zeta_1, \zeta_2 \in K$ and $x_1, x_2 \in M$ such that $\xi_i = \zeta_i \otimes x_i$ for $i = 1, 2$. Then $\zeta_1 \otimes x_1 \cdot g = \zeta_2 \otimes x_2$, which means that there exists $h \in G$ such that

$$\zeta_1 = \zeta_2 \cdot h, \quad h \cdot x_1 = x_2 \cdot g^{-1}.$$

The first equality and properness of the action of G on $\mathcal{X}$ implies that the set of possible h is finite. Then the second equality and freeness of the right action on $\mathfrak{M}$ implies that the set of possible values of g is also finite. □

Let $\mathcal{X}$ be a right G-space, and let $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ be an equivariant continuous map. If the action of G on $\mathcal{X}$ is proper, then the associated groupoid of germs of the action of G on $\mathcal{X}$ is an atlas of some orbispace $\mathcal{M}$.

Fix a basis X of the bimodule $\mathfrak{M}$ (see Definition 4). Then we have the associated action of G on X, hence we get a covering of the orbispace $\mathcal{M}$ by the orbispace $\mathcal{M}_1$ of the action

$$g(\xi, x) = (\xi \cdot g^{-1}, g(x))$$

of G on $\mathcal{X} \times \mathsf{X}$. The covering map $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is induced by the projection map $P : \mathcal{X} \times \mathsf{X} \longrightarrow \mathcal{X}$.

The semi-conjugacy $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ naturally induces a functor of the groupoids of germs, hence it defines a morphism $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ of the orbispaces. More explicitly, the functor maps the germ of the action of $g \in G$ at a point (ξ, x) to the germ of the action of $g|_x$ at $I(\xi \otimes x)$.

Definition 20. The constructed automaton $(\mathcal{M}, \mathcal{M}_1, p, \iota)$ is the *automaton associated with the G-space $\mathcal{X}$ and the semiconjugacy $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$.*

5.2. **Self-similar G-spaces from topological automata.** Suppose that $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ is a topological automaton such that the space $\mathcal{M}$ is compact, path connected and semi-locally simply connected (resp. developable, if it is an orbispace). Recall that a topological space $\mathcal{M}$ is semi-locally simply connected, if for every point $x \in \mathcal{M}$ there exists a neighborhood U such that every loop in U is homotopic in $\mathcal{M}$ to a point.

The universal covering $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is defined as the space of homotopy classes of paths starting at a fixed basepoint t. The fundamental group $\pi_1(\mathcal{M}, t)$ acts on $\widetilde{\mathcal{M}}$ in the usual way: by appending loops to the paths. The action is co-compact if $\mathcal{M}$ is compact. It is proper by semi-local simple connectedness and local compactness of $\mathcal{M}$.

The associated bimodule $\mathfrak{M}_{\mathcal{F}}$ over $\pi_1(\mathcal{M}, t)$ is the set of pairs (ℓ, z), where $z \in p^{-1}(t)$ and ℓ is a homotopy class of a path starting in t and ending in $\iota(z)$ (see [Nek05, Section 5.1.4]). The fundamental group $\pi_1(\mathcal{M}, t)$ acts on $\mathfrak{M}_{\mathcal{F}}$ on the right by appending paths

$$(\ell, z) \cdot \gamma = (\ell\gamma, z),$$

and on the left by taking lifts by p:

$$\gamma \cdot (\ell, z) = (\iota(p^{-1}(\gamma)_z)\ell, \gamma(z)),$$

where $\gamma(z)$ is the end of $p^{-1}(\gamma)_z$ (i.e., the image of z under the action of γ). Recall that in a product of paths $\ell\gamma$ the path γ is passed before ℓ.

If $\xi \in \widetilde{\mathcal{M}}$ is a point represented by a path α starting at t, and (ℓ, z) is an element of $\mathfrak{M}_{\mathcal{F}}$, then define $I(\xi \otimes (\ell, z))$ to be the point of $\widetilde{\mathcal{M}}$ represented by the path $\iota(p^{-1}(\alpha)_z)\ell$.

Proposition 5.3. *The map $I(\xi\otimes(\ell, z)) = \iota(p^{-1}(\alpha)_z)\ell$ is a well defined $\pi_1(\mathcal{M}, t)$-equivariant continuous map from $\widetilde{\mathcal{M}}\otimes\mathfrak{M}_{\mathcal{F}}$ to $\widetilde{\mathcal{M}}$.*

Proof. Equivariance and the fact that I is well defined follows directly from the definitions of the actions of $\pi_1(\mathcal{M}, t)$ on $\widetilde{\mathcal{M}}$ and $\mathfrak{M}_{\mathcal{F}}$. Continuity follows from continuity of the map ι and branches of p^{-1}. □

Proposition 5.4. *The automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ is isomorphic to the automaton associated with the $\pi_1(\mathcal{M}, t)$-space $\widetilde{\mathcal{M}}$ and the equivariant map $I : \widetilde{\mathcal{M}}\otimes\mathfrak{M}_{\mathcal{F}} \longrightarrow \widetilde{\mathcal{M}}$.*

Here two automata $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ and $\mathcal{F}' = (\mathcal{M}', \mathcal{M}'_1, p', \iota')$ are called isomorphic if there exist homeomorphisms $\psi : \mathcal{M} \longrightarrow \mathcal{M}'$ and $\psi_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}'_1$ such that $p'\circ\psi_1 = \psi\circ p$ and $\iota'\circ\psi_1 = \psi\circ\iota$.

Proof. Fix a basis $\mathsf{X} = \{x_z = (\ell_z, z) \ : \ z \in p^{-1}(t)\}$ of $\mathfrak{M}_{\mathcal{F}}$. Let (ξ, x_z) be a point of $\widetilde{\mathcal{M}}\times\mathsf{X}$ and suppose that ξ is represented by a path α. Define $\Psi_1(\xi, x_z) \in \mathcal{M}_1$ to be the end of the path $p^{-1}(\alpha)_z$. For every $\gamma\in\pi_1(\mathcal{M}, t)$ we have

$$\Psi_1(\alpha\cdot\gamma^{-1}, \gamma(z)) = \Psi_1(\alpha, z),$$

since

$$p^{-1}(\alpha\cdot\gamma^{-1})_{\gamma(z)}p^{-1}(\gamma)_z = p^{-1}(\alpha\cdot\gamma^{-1}\gamma)_z.$$

In the other direction, suppose that $\Psi_1(\xi_1, x_{z_1}) = \Psi_1(\xi_2, x_{z_2})$ and ξ_1, ξ_2 are represented by paths α_1 and α_2. Then the endpoints of the paths $p^{-1}(\alpha_1)_{z_1}$ and $p^{-1}(\alpha_2)_{z_2}$ coincide, hence the path

$$\gamma = p((p^{-1}(\alpha_1)_{z_1})^{-1}p^{-1}(\alpha_2)_{z_2}) = \alpha_1^{-1}\alpha_2$$

is an element of $\pi_1(\mathcal{M}, t)$. Then $\xi_2 = \xi_1\cdot\gamma$ and $\gamma^{-1}(z_1) = z_2$, since the path

$$(p^{-1}(\alpha_2)_{z_2})^{-1}p^{-1}(\alpha_1)_{z_1}$$

is a lift of γ^{-1} by p.

It follows that Ψ_1 induces a homeomorphism ψ_1 between the quotient of $\widetilde{\mathcal{M}}\times\mathsf{X}$ by the action $(\xi, x_z)\mapsto(\xi\cdot\gamma, x_{\gamma^{-1}(z)})$ and $\mathcal{M}_1$. It is checked now directly that this homeomorphism together with the natural homeomorphism $\psi : \widetilde{\mathcal{M}}/\pi_1(\mathcal{M}, t) \longrightarrow \mathcal{M}$ satisfies the definition of an isomorphism of automata. □

Consequently, we will not loose any automaton with connected and semi-locally simply connected base space $\mathcal{M}$, if we pass to G-spaces and equivariant maps. On the other hand, the $\mathfrak{M}$-semi-invariant G-space $\mathcal{X}$ does not have to be semi-locally simply connected, thus we can use theory of $\mathfrak{M}$-semi-invariant G-spaces for automata with more general base spaces $\mathcal{M}$.

5.3. **Iteration of automata associated with G-spaces.** Let us describe how automata associated with $\mathfrak{M}$-semi-invariant spaces are iterated.

The proof of the following lemma follows directly from the definition of tensor products.

Lemma 5.5. *Suppose that $I : \mathcal{X}\otimes\mathfrak{M} \longrightarrow \mathcal{X}$ is a G-equivariant continuous map. Then for every $n \geq 1$ the map $I^{(n)} : \mathcal{X}\otimes\mathfrak{M}^{\otimes n} \longrightarrow \mathcal{X}$ given by*

$$I^{(n)}(\xi\otimes x_1\otimes x_2\otimes\cdots\otimes x_n) = I(\ldots I(I(\xi\otimes x_1)\otimes x_2)\ldots\otimes x_n)$$

is also G-equivariant.

Let $\mathcal{X}$ be a proper co-compact right G-space and let $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ be a G-equivariant map.

Denote by $\overline{\mathcal{M}_n}$ the orbispace $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}/G$. Denote by $\mathcal{M}_n$ the orbispace of the action of G on $\mathcal{X} \times \mathsf{X}^n$ given by

$$g \cdot (\xi, v) = (\xi \cdot g^{-1}, g(v)).$$

Proposition 5.6. *The map $(\xi, v) \mapsto \xi \otimes v$ induces a homeomorphism of the underlying space of $\mathcal{M}_n$ with the underlying space of $\overline{\mathcal{M}_n}$.*

Proof. If the points (ξ_1, v_1) and (ξ_2, v_2) belong to one orbit of the atlas of $\mathcal{M}_n$ then there exists $g \in G$ such that $(\xi_1, v_1) = (\xi_2 \cdot g^{-1}, g(v_2))$. Then

$$\xi_2 \otimes v_2 = \xi_2 \cdot g^{-1} \otimes g \cdot v_2 = \xi_2 \cdot g^{-1} \otimes g(v_2) \cdot g|_{v_2} = \xi_1 \otimes v_1 \cdot g|_{v_2},$$

i.e., $\xi_1 \otimes v_1$ and $\xi_2 \otimes v_2$ belong to one G-orbit.

On the other hand, if there exists $h \in G$ such that $\xi_2 \otimes v_2 = \xi_1 \otimes v_1 \cdot h$, then there exists g such that $\xi_2 = \xi_1 \cdot g$ and $g \cdot v_2 = v_1 \cdot h$, by the definition of a tensor product. Then $v_1 = g(v_2)$ and

$$(\xi_1, v_1) = (\xi_2 \cdot g^{-1}, g(v_2)),$$

i.e., (ξ_1, v_1) and (ξ_2, v_2) belong to one orbit of the atlas of $\mathcal{M}_n$. $\square$

Even though the underlying spaces of $\mathcal{M}_n$ and $\overline{\mathcal{M}_n}$ are homeomorphic, the orbispaces might be different. The isotropy groups of $\overline{\mathcal{M}_n}$ are quotients of the corresponding isotropy groups of $\mathcal{M}_n$.

The following proposition follow directly from the definitions, see also Proposition 4.1.

Proposition 5.7. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ be the automaton associated with a proper right G-space $\mathcal{X}$ and a G-equivariant map $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$.*

Then the nth iteration of $\mathcal{F}$ is the automaton

$$\mathcal{F}^{\circ n} = (\mathcal{M}, \mathcal{M}_n, p_0 \circ \cdots \circ p_n, \iota_0 \circ \cdots \circ \iota_n)$$

associated with the space $\mathcal{X}$ and the semiconjugacy $I^{(n)} : \mathcal{X} \otimes \mathfrak{M}^{\otimes n} \longrightarrow \mathcal{X}$.

The covering $p_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is induced by the correspondence

$$\xi \otimes v \otimes x \mapsto \xi \otimes v,$$

for $v \in \mathfrak{M}^{\otimes n}$ and $x \in \mathfrak{M}$.

The map $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is induced by the map $I_n : \mathcal{X} \otimes \mathfrak{M}^{\otimes(n+1)} \longrightarrow \mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ given by

$$I_n(\xi \otimes x \otimes v) = I(\xi \otimes x) \otimes v$$

for $x \in \mathfrak{M}$ and $v \in \mathfrak{M}^{\otimes n}$.

The proof of the following proposition is the same as the proof of Theorem 5.3.1 in [Nek05].

Proposition 5.8. *Suppose that $\mathcal{X}$ is a path-connected proper right G-space and suppose that $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ is a G-equivariant continuous map. Then the iterated monodromy group of the associated topological automaton coincides with the faithful quotient of the self-similar group G.*

5.4. **Contracting self-similarities.**

Definition 21. Suppose that $\mathcal{X}$ is a metric space and G acts on it by isometries, so that the action is proper and co-compact. We say that an equivariant map $I:\mathcal{X}\otimes\mathfrak{M}\longrightarrow\mathcal{X}$ is *contracting* if there exist n and $0<\lambda<1$ such that

$$d(I^{(n)}(\xi_1\otimes v), I^{(n)}(\xi_2\otimes v))\le\lambda d(\xi_1,\xi_2)$$

for all $\xi_1,\xi_2\in\mathcal{X}$ and $v\in\mathfrak{M}^{\otimes n}$.

The maps $I^{(n)}$ are defined in Lemma 5.5.

Theorem 5.9. *Let (G,X) be a contracting group and let $\mathfrak{M}$ be the associated permutational G-bimodule. Suppose that $\mathcal{X}$ is a locally compact metric space with a co-compact proper right G-action by isometries and let $I:\mathcal{X}\otimes\mathfrak{M}\longrightarrow\mathcal{X}$ be a contracting equivariant map. Then the projective limit of the G-spaces and the G-equivariant maps*

$$\mathcal{X}\xleftarrow{I_0}\mathcal{X}\otimes\mathfrak{M}\xleftarrow{I_1}\mathcal{X}\otimes\mathfrak{M}^{\otimes 2}\xleftarrow{I_2}\mathcal{X}\otimes\mathfrak{M}^{\otimes 3}\xleftarrow{I_3}\cdots,$$

is homeomorphic as a G-space to the limit G-space $\mathcal{X}_G$.

For every $x\in\mathfrak{M}$ the maps

$$(\xi\otimes v)\mapsto(\xi\otimes v\otimes x):\mathcal{X}\otimes\mathfrak{M}^{\otimes n}\longrightarrow\mathcal{X}\otimes\mathfrak{M}^{\otimes(n+1)}$$

agree with the maps I_n and their limit is the map $\xi\mapsto\xi\otimes x$ on $\mathcal{X}_G$.

The maps $I_n:\mathcal{X}\otimes\mathfrak{M}^{\otimes(n+1)}\longrightarrow\mathcal{X}\otimes\mathfrak{M}^{\otimes n}$ were defined in Proposition 5.7.

Proof. Let $K_0\subset\mathcal{X}$ be a compact set such that $\bigcup_{g\in G}K_0\cdot g=\mathcal{X}$. Choose a basis X of $\mathfrak{M}$. There exists a compact set $K\supseteq K_0$ such that for every $x\in\mathsf{X}$ and $\xi\in K$ we have $I(\xi\otimes x)\in K$. One can take, for instance, the closure of the set of points of the form $I^{(n)}(\xi\otimes v)$ for $\xi\in K_0$ and $v\in\mathsf{X}^n$, which has finite diameter, by contraction of I.

Every point of $\mathcal{X}\otimes\mathfrak{M}^n$ can be written in the form $\xi\otimes v\cdot g$ for $\xi\in K$, $v\in\mathsf{X}^n$ and $g\in G$.

Hence, every point ζ of the inverse limit is represented by a sequence

$$\xi_0\cdot g,\quad \xi_1\otimes x_1\cdot g,\quad \xi_2\otimes x_2x_1\cdot g,\quad \ldots,\quad \xi_n\otimes x_n\ldots x_2x_1\cdot g,$$

for some $g\in G$, $x_i\in\mathsf{X}$ and $\xi_n\in K$ such that $I(\xi_n\otimes x_n)=\xi_{n-1}$ for all $n\ge 1$.

Let us put into correspondence to ζ the point $L(\zeta)\in\mathcal{X}_G$ represented by the sequence $\ldots x_2x_1\cdot g$.

Let us show that the map L is well defined. Suppose that we have the same point ζ of the inverse limit is represented in two different ways:

$$\xi_0\cdot g=\eta_0\cdot h,\quad \xi_1\otimes x_1\cdot g=\eta_1\otimes y_1\cdot h,\quad \xi_2\otimes x_2x_1\cdot g=\eta_2\otimes y_2y_1\cdot h,\quad\ldots$$

for $\xi_n,\eta_n\in K$, $x_n,y_n\in\mathsf{X}$ and $g,h\in G$. Then there exists a sequence $g_n\in G$ such that

$$\xi_n=\eta_n\cdot g_n,\qquad g_n\cdot x_n\ldots x_2x_1\cdot g=y_n\ldots y_2y_1\cdot h$$

for all $n\ge 0$. (Notation is explained after Proposition 2.3.) By compactness of K and properness of the action of G on $\mathcal{X}$, the set of possible values of the sequence g_n is finite, hence the sequences $\ldots x_2x_1\cdot g$ and $\ldots y_2y_1\cdot h$ are asymptotically equivalent and represent the same point of $\mathcal{X}_G$.

Let us show that every point of $\mathcal{X}_G$ is equal to $L(\zeta)$ for some point ζ of the inverse limit of the spaces $\mathcal{X}\otimes\mathfrak{M}^{\otimes n}$ (i.e., that the map L is onto). Let $\ldots x_2x_1\cdot g$

be an arbitrary point of the limit G-space $\mathcal{X}_G$. For every n chose an arbitrary point $\xi_{n,n} \in K$ and consider for $k = 1, \ldots, n-1$ the points defined inductively as $\xi_{n,k-1} = I(\xi_{n,k} \otimes x_k)$. Then the sequence

$$\xi_{n,0} \cdot g, \quad \xi_{n,1} \otimes x_1 \cdot g, \quad \ldots \quad \xi_{n,n} \otimes x_n \ldots x_2 x_1 \cdot g$$

agrees with the maps in the inverse sequence of the spaces $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}$. We can choose, by compactness of K, an increasing sequence $n_{k,1}$ such that the sequence $\xi_{n_{k,1},0}$ converges to a point ξ_0. Then we can choose a subsequence $n_{k,2}$ of $n_{k,1}$ such that $\xi_{n_{k,2},1}$ converges to a point ξ_1, etc. In the limit, by continuity of I, we get a sequence

$$\xi_0 \cdot g, \quad \xi_1 \otimes x_1 \cdot g, \quad \ldots, \quad \xi_n \otimes x_n \ldots x_2 x_1 \cdot g, \quad \ldots$$

representing a point ζ of the inverse limit such that $L(\zeta) = \ldots x_2 x_1 \cdot g$.

Let us show that L is a one-to-one map. Suppose that we have two sequences

$$\xi_0 \cdot g, \quad \xi_1 \otimes x_1 \cdot g, \quad \xi_2 \otimes x_2 x_1 \cdot g, \quad \ldots$$

and

$$\eta_0 \cdot h, \quad \eta_1 \otimes y_1 \cdot h, \quad \eta_2 \otimes y_2 y_1 \cdot h, \quad \ldots$$

for $\xi_n, \eta_n \in K$, $x_n, y_n \in \mathsf{X}$ and $g, h \in G$ such that the sequences $\ldots x_2 x_1 \cdot g$ and $\ldots y_2 y_1 \cdot h$ are asymptotically equivalent. Let $g_n \in G$ be the sequence implementing the asymptotic equivalence, i.e., a sequence with a finite set A of values such that $g_n \cdot x_n \ldots x_2 x_1 \cdot g = y_n \ldots y_2 y_1 \cdot h$. Then the second point of the inverse limit is written as

$$\eta_0 \cdot g_0 g, \quad \eta_1 \cdot g_1 \otimes x_1 \cdot g, \quad \eta_2 \cdot g_2 \otimes x_2 x_1 \cdot g, \quad \ldots.$$

We have $I^{(n)}(\xi_n \otimes x_n \ldots x_2 x_1 \cdot g) = \xi_0 \cdot g$ and $I^{(n)}(\eta_n \cdot g_n \otimes x_n \ldots x_2 x_1 \cdot g) = \eta_0 \cdot g_0 g$. The points ξ_n and $\eta_n \cdot g_n$ belong to the set $K \cup K \cdot A$ of finite diameter. This implies, by contraction of I that $\xi_0 = \eta_0 \cdot g_0$. One proves in the same way that $\xi_n = \eta_n \cdot g_n$ for all n, i.e., that the two points of the inverse limit are the same.

Continuity of the map L^{-1} (i.e., that sequences with long common beginnings correspond to close points of the inverse limit) follows directly from the contraction property for I.

The maps $I^{(n)} = I_1 \circ \cdots \circ I_n : \mathcal{X} \otimes \mathfrak{M}^{\otimes n} \longrightarrow \mathcal{X}$ are proper by Lemmata 5.2 and 5.1, which implies that the inverse limit is locally compact.

We have constructed an equivariant continuous bijection between the inverse limit of the spaces $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ and $\mathcal{X}_G$. Since both spaces are locally compact and Hausdorff, this map is a homeomorphism.

The statement about the map $\xi \mapsto \xi \otimes x$ follows directly from the construction of the homeomorphism L. □

We see that if the equivariant map $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ is contracting, then the spaces $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ are approximations of the limit G-space $\mathcal{X}_G$, hence the spaces $\mathcal{M}_n$ are approximations of the limit space $\mathcal{J}_G$ of the group G.

5.5. **Contracting topological automata.**

Definition 22. Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton such that $\mathcal{M}$ is a compact, path connected, and semi-locally path connected (orbi)space. We say that the topological automaton $\mathcal{F}$ is *contracting* if there exists a length structure on $\mathcal{M}$ and $\lambda < 1$ such that for every rectifiable path γ in $\mathcal{M}_1$ the path $\iota(\gamma)$ has length less than λ times the length of γ (with respect to the length structure on $\mathcal{M}_1$ equal to the pull back by f of the length structure on $\mathcal{M}$)

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ be a contracting automaton. Let $\widetilde{\mathcal{M}}$ be the universal covering of $\mathcal{M}$, let $\mathfrak{M}_{\mathcal{F}}$ be the associated $\pi_1(\mathcal{M})$-bimodule, and let $I : \widetilde{\mathcal{M}} \otimes \mathfrak{M}_{\mathcal{F}} \longrightarrow \widetilde{\mathcal{M}}$ be the self-similarity defined in Subsection 5.2. Then $\widetilde{\mathcal{M}}$ has a natural length structure, which is the lift of the length structure on $\mathcal{M}$ (length of a curve in $\widetilde{\mathcal{M}}$ is equal to the length of its image in $\mathcal{M}$). It follows then from the definition of the map I and Definition 22 that I is contracting. This implies that the iterated monodromy group of $\mathcal{F}$ is a contracting self-similar group. We get now the following corollary of Theorem 5.9 and Proposition 5.7.

Theorem 5.10. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a contracting topological automaton with semi-locally simply connected orbispace $\mathcal{M}$. Then the iterated monodromy group* $\mathrm{IMG}\,(\mathcal{F})$ *is contracting and the system* $(\lim_\iota \mathcal{F}, f_\infty)$ *is topologically conjugate to the limit dynamical system* $(\mathcal{J}_{\mathrm{IMG}(\mathcal{F})}, \mathsf{s})$ *of the iterated monodromy group.*

This means that if a topological automaton $\mathcal{F} = (\mathcal{M}_1, \mathcal{M}, f, \iota)$ is contracting, then the spaces $\mathcal{M}_n$ can be used as approximations of the limit space $\mathcal{J}_{\mathrm{IMG}(\mathcal{F})}$. The natural maps $\pi_n : \lim_\iota \mathcal{F} \longrightarrow \mathcal{M}_n$ will become more and more "precise" in the sense that the difference $\mathcal{M}_n \setminus \pi_n(\lim_\iota \mathcal{F})$ and the fibers $\pi_n^{-1}(x)$ become "smaller".

Corollary 5.11. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are combinatorially equivalent contracting automata, then the dynamical systems* $(\lim_\iota \mathcal{F}_1, f_\infty)$ *and* $(\lim_\iota \mathcal{F}_2, f_\infty)$ *are topologically conjugate.*

5.6. **Homotopy equivalence and contracting automata.** Contracting topological automata can be simplified using the following general procedure.

Proposition 5.12. *Let (G, X) be a contracting group and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be metric spaces with proper co-compact right actions of G by isometries. Suppose that there exists a contracting G-equivariant map $I : \mathcal{X}_1 \otimes \mathfrak{M} \longrightarrow \mathcal{X}_1$ and Lipschitz G-equivariant maps $F_1 : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $F_2 : \mathcal{X}_2 \longrightarrow \mathcal{X}_1$. Then there exists $m \geq 1$ and a contracting G-equivariant map $\Psi : \mathcal{X}_2 \otimes \mathfrak{M}^{\otimes m} \longrightarrow \mathcal{X}_2$.*

Proof. For any given m consider the map $\Psi : \mathcal{X}_2 \otimes \mathfrak{M}^{\otimes m} \longrightarrow \mathcal{X}_2$ defined by the equality

$$\Psi(\xi \otimes v) = F_1 \left(I_1^{(m)}(F_2(\xi) \otimes v) \right).$$

It is easy to see that it is G-equivariant and well defined (the latter means that $F_2(\xi) \otimes v$ depends only on $\xi \otimes v$). If F_1 and F_2 are Lipschitz with coefficient L and $I^{(m)}$ is contracting with coefficient $\lambda < L^{-2}$, then Ψ is contracting. □

Corollary 5.13. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a contracting topological automaton such that $\mathcal{M}$ is a finite simplicial complex (or complex of groups) with a piecewise Riemannian length structure. Then for any finite simplicial complex (resp. complex of groups) $\mathcal{M}'$ homotopically equivalent to $\mathcal{M}$ there exists n and a contracting automaton $\mathcal{F}' = (\mathcal{M}', \mathcal{M}'_1, f', \iota')$ combinatorially equivalent to $\mathcal{F}^{\circ n}$.*

Proof. Homotopy equivalence will lift to a pair of $\pi_1(\mathcal{M}) \cong \pi_1(\mathcal{M}')$-equivariant maps between the universal coverings of $\mathcal{M}$. By Simplicial Approximation Theorem, we may assume that these maps are simplicial, hence Lipschitz for some length structure on $\mathcal{M}'$. □

6. Simplicial approximations of the limit spaces

6.1. **Topological nucleus.** Let (G, X) be a contracting self-similar group with nucleus $\mathcal{N}$. The aim of this section is to find a simple construction of a proper co-compact G-space $\mathcal{X}$ and a contracting equivariant map $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$.

We assume for simplicity that the group G is finitely generated, and the action (G, X) is self-replicating, i.e., that the left action of G on the self-similarity bimodule $\mathfrak{M} = \mathsf{X} \cdot G$ is transitive.

Defining an equivariant continuous map $I : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ is equivalent to defining a family of continuous maps $I_x : \mathcal{X} \longrightarrow \mathcal{X} : \xi \mapsto I(\xi \otimes x)$ for all $x \in \mathfrak{M}$ satisfying the conditions

$$I_x(\xi \cdot g) = I_{g \cdot x}(\xi), \qquad I_{x \cdot g}(\xi) = I_x(\xi) \cdot g, \tag{6}$$

for all $x \in \mathfrak{M}$, $g \in G$ and $\xi \in \mathcal{X}$. The first condition is equivalent to the condition for the map $I(\xi \otimes x) = I_x(\xi)$ to be well defined. The second condition is equivalent to equivariance of I.

The iteration $I^{(n)} : \mathcal{X} \otimes \mathfrak{M}^{\otimes n} \longrightarrow \mathcal{X}$ is defined then by the compositions $I_{x_1 \otimes \cdots \otimes x_n}(\xi) = I_{x_1} \circ \cdots \circ I_{x_n}(\xi)$.

Note that it is enough to define the maps I_x for $x \in \mathsf{X}$, since every element of $\mathfrak{M}$ can be written as $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$. Then the only condition to check is

$$I_x(\xi \cdot g) = I_{g(x)}(\xi) \cdot g|_x \tag{7}$$

for all $x \in \mathsf{X}$ and $g \in G$.

Moreover, since we assume that the action is self-replicating, every element of $\mathfrak{M}$ can be written as $g \cdot x$ for a fixed $x \in \mathfrak{M}$. Then it is enough to define one map I_x satisfying condition (7) for all g in the stabilizer of x. This coincides with condition (1) from Introduction.

The simplest example of a $\mathfrak{M}$-semi-invariant G-space is the group G itself with respect to the action by right translations and the maps

$$I_x(g) = g|_x. \tag{8}$$

Then condition (7) follows directly from the basic properties (2) of sections.

Next natural construction will be to choose a finite generating set $S = S^{-1}$ and consider the corresponding Rips complex, i.e., the simplicial complex $\Gamma(G, S)$ with the set of vertices G and the set of simplices equal to the set of subsets $A \subset G$ such that $gh^{-1} \in S$ for all $g, h \in A$. If S is *self-similar*, i.e., if $S|_x \subset S$ for all $x \in \mathsf{X}$, then the maps I_x, defined by (8), are simplicial and define a G-equivariant map $I : \Gamma(G, S) \otimes \mathfrak{M} \mapsto \Gamma(G, S)$.

For every finite self-similar set S there exists n such that $S_n = \bigcup_{v \in \mathsf{X}^n} S|_v$ is a subset of the nucleus $\mathcal{N}$. Then $I^{(n)}(\Gamma(G, S_n) \otimes \mathfrak{M}^{\otimes n})$ is a subcomplex of $\Gamma(G, \mathcal{N})$. Consequently, it is sufficient to consider just the case $S = \mathcal{N}$.

Moreover, it may happen that $I(\Gamma(G, \mathcal{N}) \otimes \mathfrak{M})$ is a proper sub-complex of $\Gamma(G, \mathcal{N})$, and we can then pass to a smaller complex. Namely, we get a decreasing sequence of simplicial complexes

$$\Gamma(G, \mathcal{N}) \supseteq I(\Gamma(G, \mathcal{N}) \otimes \mathfrak{M}) \supseteq I^{(2)}(\Gamma(G, \mathcal{N}) \otimes \mathfrak{M}^{\otimes 2}) \supseteq \cdots,$$

which has to stabilize, since all these complexes are G-invariant, and $\Gamma(G, \mathcal{N})$ is locally finite.

Let us describe the complex $\Gamma = \bigcap_{n \geq 0} I^{(n)}(\Gamma(G, \mathcal{N}) \otimes \mathfrak{M}^{\otimes n})$. Since it is G-invariant, it is sufficient to describe the set of simplices containing the identity.

Proposition 6.1. *A subset $A \subset \mathcal{N}$ containing the identity element is a simplex of Γ if and only if there exists a sequence $\ldots x_2x_1 \in \mathsf{X}^{-\omega}$ and a sequence $A_n \subset \mathcal{N}$ such that $A_0 = A$ and $A_n|_{x_n} = A_{n-1}$ for all $n \geq 1$.*

Proof. A subset $A \subset G$ is a simplex of Γ if and only if there exists a sequence B_n of simplices of $\Gamma(G, \mathcal{N})$ and a sequence of words $v_n \in \mathsf{X}^n$ such that $B_n|_{v_n} = A$.

If A_n and $\ldots x_2x_1 \in \mathsf{X}^{-\omega}$ satisfy the conditions of the proposition, then we can take $B_n = A_n$ and $v_n = x_n \ldots x_2x_1$, and conclude that A is a simplex of Γ.

Let us prove the other direction of the proposition. Let $\mathcal{A}_n$ be the set of simplices A_n of $\Gamma(G, \mathcal{N})$ containing the identity for which there exists $x_n \ldots x_2x_1 \in \mathsf{X}^n$ such that $A_n|_{x_n \ldots x_2x_1} = A$. Note that in this case we have $A_n|_{x_n} \in \mathcal{A}_n$, so that we get a sequence of maps $A_n \mapsto A_n|_{x_n}$ from $\mathcal{A}_n$ to $\mathcal{A}_{n-1}$. The sets $\mathcal{A}_n$ are finite, and every element of the inverse limit of the sets $\mathcal{A}_n$ with respect to the described maps gives us sequence $(A_n)_{n\geq 1}$ and $\ldots x_2x_1 \in \mathsf{X}^{-\omega}$ satisfying the conditions of the proposition. It remains hence to prove that the sets $\mathcal{A}_n$ are not empty.

If a simplex B_n of $\Gamma(G, \mathcal{N})$ and a word $v \in \mathsf{X}^n$ are such that $B_n|_v = A$, then there exists $h \in B_n$ such that $h|_v = 1$. Then $B_n \cdot h^{-1}$ is a simplex of $\Gamma(G, \mathcal{N})$ containing the identity and such that $(B_n \cdot h^{-1})|_{h(v)} = B_n|_v \cdot (h|_v)^{-1} = A$, i.e., $B_n \cdot h^{-1}$ is an element of $\mathcal{A}_n$. □

As a direct corollary of Proposition 6.1 we get a more explicit description of the simplices of Γ.

Corollary 6.2. *Denote by $\mathcal{B}$ the set of subsets of $\mathcal{N}$ containing the identity. Construct a directed graph with the set of vertices $\mathcal{B}$ in which a there is an arrow starting at a vertex A_1 and ending in a vertex A_2 if there exists $x \in \mathsf{X}$ such that $A_2 = A_1|_x$.*

A set $A \in \mathcal{B}$ is a simplex of Γ if and only if it is an end of a directed path starting in a directed cycle in the constructed graph.

Proposition 6.3. *For every finite subset $A \subset G$ there exists n such that for all words $v \in \mathsf{X}^*$ of length at least n the set $A|_v$ is a simplex of Γ.*

Proof. Replacing A by $A \cdot g^{-1}$ for some $g \in A$, we may assume that A contains the identity. There exists n_1 such that $A|_v \subset \mathcal{N}$, by definition of the nucleus. Then, by Corollary 6.2, there exists n_2 such that $A|_v|_u$ is a simplex of Γ for all words u of length at least n_2. The number $n = n_1 + n_2$ satisfies the conditions of the proposition. □

The simplices of Γ have the following geometric description.

Proposition 6.4. *A subset $A \subset G$ is a simplex of Γ if and only if there exists a point ξ of $\mathcal{X}_G$ such that for every $g \in A$ the point ξ can be represented by $\ldots x_2x_1 \cdot g \in \mathsf{X}^{-\omega} \times G$.*

In other words, A is a simplex of Γ if and only if the intersection

$$T_A = \bigcap_{g \in A} \mathcal{T} \cdot g$$

is non-empty.

Proof. Suppose that for every $g \in A$ there exists a sequence $\ldots x_2x_1 \cdot h \in \mathsf{X}^{-\omega} \times G$ representing a point of $\bigcap_{g \in A} \mathcal{T} \cdot g$, i.e., equivalent to some sequences $w_g = \ldots y_2y_1 \cdot g$.

For every w_g there exists then a sequence $h_{n,g} \in \mathcal{N}$ such that $h_{n,g} \cdot x_n = y_n \cdot h_{n-1,g}$ and $h_0 h = g$. Denote $A_n = \{h_{n,g}\}_{g \in A}$. Then $A_n|_{x_n} = A_{n-1}$ and $A_0 = A \cdot h^{-1}$, which implies that A is a simplex of Γ.

In the other direction, if A_n and $\ldots x_2 x_1$ are such that $A_n|_{x_n} = A_{n-1}$ and $A_0 = A \cdot h^{-1}$ for some $h \in G$, then for every $g \in A$ there exists a sequence $h_{n,g} \in A_n$ such that $h_{n,g}|_{x_n} = h_{n-1,g}$ and $h_{0,g} h = g$. In this case the sequence $\ldots x_2 x_1 \cdot h$ is equivalent to the sequences $\ldots h_{2,g}(x_2) h_{1,g}(x_1) \cdot g$, i.e., the corresponding point of $\mathcal{X}_G$ belongs to every tile $\mathcal{T} \cdot g$, $g \in A$. $\square$

In view of Proposition 6.4 we introduce the following terminology.

Definition 23. The complex Γ is called the *tiling nerve* of (G, X). The simplices of Γ are called *adjacency subsets* of G.

Denote by Γ_n the G-space $\Gamma \otimes \mathfrak{M}^{\otimes n}$. Recall that the maps $I_v : \Gamma \longrightarrow \Gamma$, for $v \in \mathsf{X}^n$, defining the equivariant maps $I^{(n)} : \Gamma_n \longrightarrow \Gamma$ are simplicial maps given by

$$I_v(g) = g|_v.$$

Definition 24. Denote by $J_n(G)$ the quotient Γ_n / G (in particular, $J_0(G)$ is Γ / G). The topological automaton $(J_0(G), J_1(G), p, \iota)$ associated with the map $I : \Gamma \otimes \mathfrak{M} \longrightarrow \Gamma$ is called the *topological nucleus* of the group (G, X).

Recall that the covering $p : J_1(G) \longrightarrow J_0(G)$ is induced by the correspondence $\xi \otimes x \mapsto \xi$ from $\Gamma \otimes \mathfrak{M}$ to Γ, the map $\iota : J_1(G) \longrightarrow J_0(G)$ is induced by the map $I : \Gamma \otimes \mathfrak{M} \longrightarrow \Gamma$.

Note that the restriction of the topological nucleus onto its one-skeleton coincides with the dual Moore diagram of the nucleus of G. In particular, the one-skeleton of the complex $J_n(G)$ is the Schreier graph of the action of G on the nth level of the tree X^*.

6.2. **Recurrent description of $J_n(G)$.** The spaces $J_n(G)$ can be constructed by the following simple cut-and-paste procedure.

The barycentric subdivision Γ' of Γ is isomorphic, as a simplicial complex, to the realization of the poset (with respect to inclusion) of the adjacency subsets of G.

Let us take as a fundamental domain of the G-action on Γ the union T_0 of the simplices of the barycentric subdivision Γ' containing $1 \in G$.

The set of vertices of T_0 is the set $\mathcal{A}$ of adjacency subsets of G containing the identity. The complex T_0 is isomorphic to the geometric realization of the poset $\mathcal{A}$ with respect to inclusion.

For every $g \in \mathcal{N} \setminus \{1\}$ consider the subset $\mathcal{A}_g$ of the poset $\mathcal{A}$ consisting of the adjacency sets containing g. It is a sub-poset of $\mathcal{A}$ and it is equal to the intersection of $\mathcal{A}$ with $\mathcal{A} \cdot g = \{A \cdot g \ : \ A \in \mathcal{A}\}$. Let $K_{g,0}$ be the corresponding sub-complex of T_0. For every $A \in \mathcal{A}_g$ the set $A \cdot g^{-1}$ belongs to $\mathcal{A}_{g^{-1}}$, since $A \cdot g^{-1} \supset \{1, g\} \cdot g^{-1} = \{g^{-1}, 1\}$.

The map $A \mapsto A \cdot g^{-1}$ is an order-preserving bijection from $\mathcal{A}_g$ to $\mathcal{A}_{g^{-1}}$. Denote by $\kappa_{g,0} : K_{g,0} \longrightarrow K_{g^{-1},0}$ the corresponding isomorphism of the sub-complexes. It coincides with the restriction of the map $\xi \mapsto \xi \cdot g^{-1}$ onto $T_0 \cap T_0 \cdot g = K_{g,0}$.

It follows then directly from the definitions that the complex $J_0(G)$ is the quotient of T_0 by the identifications κ_g for all $g \in \mathcal{N}$.

Proposition 6.5. *Define the complex T_n inductively as the quotient of $T_{n-1} \times \mathsf{X}$ by the identifications*

$$(\xi, x) \sim (\kappa_{g,n-1}(\xi), g(x))$$

for all $\xi \in K_{g,n-1}$, $x \in \mathsf{X}$ and $g \in \mathcal{N} \setminus \{1\}$ such that $g|_x = 1$.

Define the identification $\kappa_{g,n}$ on the image $K_{g,n}$ of the set

$$\bigcup_{x \in \mathsf{X}, h \in \mathcal{N}, h|_x = g} (K_{h,n-1}, x) \subset T_{n-1} \times \mathsf{X}$$

in T_n and acting by the rule

$$\kappa_{g,n} : (\xi, x) \mapsto (\kappa_{h,n-1}(\xi), h(x)),$$

where $h \in \mathcal{N}$ is such that $h|_x = g$.

Then the complex $J_n(G)$ is isomorphic to the quotient of T_n by the identifications $\kappa_{g,n}$.

The covering $p_n : J_{n+1}(G) \longrightarrow J_n(G)$ is induced by the map $(\xi, x) \mapsto \xi$ for $\xi \in J_n(G)$ and $x \in \mathsf{X}$. The map $\iota_n : J_{n+1}(G) \longrightarrow J_n(G)$ is induced by the map $(\xi, x) \mapsto (\iota_{n-1}(\xi), x)$ for $\xi \in J_n(G)$ and $x \in \mathsf{X}$.

Proof. Direct corollary of the definition of the tensor product $\Gamma \otimes \mathfrak{M}^{\otimes n}$ and Proposition 5.7. Here T_n is the fundamental domain $\bigcup_{v \in \mathsf{X}^n} T_0 \otimes v$ of the action of G on $\Gamma \otimes \mathfrak{M}^{\otimes n}$. □

The recursive rule described in Proposition 6.5 is conveniently encoded by the dual Moore diagram of the nucleus. Recall that it this diagram the vertices are the letters of the alphabet X, and for every $g \in \mathcal{N} \setminus \{1\}$ there is an arrow labeled by $(g, g|_x)$ starting at x and ending in $g(x)$.

We can interpret now the arrows of the dual Moore diagram of the nucleus as instructions how to paste together the copies (T_{n-1}, x) of T_{n-1} into the complex T_n, and the identifications $\kappa_{g,n-1}$ into the identifications $\kappa_{g,n}$.

Namely, every vertex x corresponds to the piece (T_{n-1}, x). The arrows labeled by $(g, 1)$ describe how to paste together the complexes (T_{n-1}, x) into T_n: one has to take the piece corresponding to the beginning of the arrow and attach it by the map $\kappa_{g,n-1}$ to the piece corresponding to the end of the arrow.

Every arrow labeled by (h, g) will describe the part of the identification rule $\kappa_{g,n}$ that maps (ξ, x) to $(\kappa_{h,n-1}(\xi), y)$, where x is the beginning and y is the end of the arrow.

6.3. **Topological nucleus as a contracting automaton.** The maps $I^{(n)} : \Gamma_n \longrightarrow \Gamma$ are not contracting, since there always exist simplices S of Γ_n mapped isometrically by I_v for some $v \in \mathsf{X}^n$. Nevertheless, we can transform $I^{(n)}$ into a contracting map.

Theorem 6.6. *There exists $n \geq 1$ such that the map $I^{(n)} : \Gamma \otimes \mathfrak{M}^{\otimes n} \longrightarrow \Gamma$ is G-equivariantly homotopic (i.e., is homotopic through equivariant maps) to a contracting map.*

Proof. By Proposition 6.3, there exists n such that, for any adjacency set A, the set $(\mathcal{N} \cdot A)|_v$ is an adjacency set for all words $v \in \mathsf{X}^*$ of length at least n. Fix such a number n and define then, for $g \in G$ and $v \in \mathsf{X}^n$, the point $\widetilde{I}_v(g)$ as the

barycenter of the simplex $(\mathcal{N} \cdot g)|_v$. For every $h, g \in G$ and $v \in \mathsf{X}^n$ we have $(\mathcal{N} \cdot h \cdot g)|_v = (\mathcal{N} \cdot h)|_{g(v)} \cdot g|_v$, hence

$$\widetilde{I}_v(h \cdot g) = \widetilde{I}_{g(v)}(h) \cdot g|_v,$$

i.e., condition (7) is satisfied.

Let $v \in \mathsf{X}^n$ be an arbitrary word, and let $A \subset G$ be an adjacency set, i.e., a simplex of Γ. For every $g \in A$ we have

$$g|_v \in (\mathcal{N} \cdot g)|_v \subset (\mathcal{N} \cdot A)|_v,$$

hence all the simplices $(\mathcal{N} \cdot g)|_v$ for $g \in A$ belong to the simplex $\Delta = (\mathcal{N} \cdot A)|_v$. Consequently, we can linearly extend inside Δ the map $\widetilde{I}_v$ from the set of vertices of the simplex A to the whole geometric realization of A. These extensions agree with each other, satisfy (7), and hence define a G-equivariant continuous map $\widetilde{I} : \Gamma_n \longrightarrow \Gamma$. The points $I_v(g) = g|_v$ also belong to the simplex Δ, hence the convex combination $(1-t)I^{(n)} + t\widetilde{I}$ inside Δ is a G-equivariant homotopy from $I^{(n)}$ to $\widetilde{I}$.

Fix some $g_0 \in A$. For every $g \in A$ we have $g_0 g^{-1} \in \mathcal{N}$, hence the simplices $(\mathcal{N} \cdot g)|_v$ contain $g_0|_v$ for all $g \in A$. Consequently, their barycenters $\widetilde{I}_v(g)$ are contained in the image of Δ under the homothety with center in $I_v(g_0)$ and coefficient $\dim \Delta/(\dim \Delta + 1)$. The maps $\widetilde{I}_v$ for $v \in \mathfrak{M}^{\otimes n}$ are affine on the simplices of Γ, hence we get exponential decreasing of the diameters of the simplices under compositions of the maps $\widetilde{I}_v$, for $v \in \mathsf{X}^n$, which in turn implies that $\widetilde{I}$ is contracting. $\square$

Even though we have proved that only some iteration $(J_0(G), J_n(G), p^n, \iota^n)$ of the topological nucleus $(J_0(G), J_1(G), p, \iota)$ is homotopic to a contracting automaton, one can use this fact to approximate not only the limit space $\mathcal{J}_G$, but also the limit dynamical system $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ (and not just its nth iteration).

Let us introduce some notation. Let $\widetilde{I} : \Gamma_n \longrightarrow \Gamma$ be a contracting map equivariantly homotopic to the map $I^{(n)} : \Gamma_n \longrightarrow \Gamma$, as in Theorem 6.6. Denote by $\widetilde{\iota}_k : J_{n(k+1)}(G) \longrightarrow J_{nk}(G)$ the map induced by

$$\xi \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto \widetilde{I}(\xi \otimes v_1) \otimes v_2 \otimes \cdots \otimes v_k,$$

for $v_i \in \mathfrak{M}^{\otimes n}$; by $\widetilde{\iota}'_k : J_{n(k+1)+1}(G) \longrightarrow J_{nk+1}(G)$ the map induced by

$$\xi \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes x \mapsto \widetilde{I}(\xi \otimes v_1) \otimes v_2 \otimes \cdots \otimes v_k \otimes x,$$

for $v_i \in \mathfrak{M}^{\otimes n}$ and $x \in \mathfrak{M}$; by $\iota_{nk} : J_{nk+1}(G) \longrightarrow J_{nk}(G)$ and $p_{nk} : J_{nk+1}(G) \longrightarrow J_{nk}(G)$, as before, the maps induced by

$$\xi \otimes x \otimes v_1 \otimes \cdots \otimes v_k \mapsto I(\xi \otimes x) \otimes v_1 \otimes \cdots \otimes v_k,$$

and

$$\xi \otimes v_1 \otimes \cdots \otimes v_k \otimes x \mapsto \xi \otimes v_1 \otimes \cdots \otimes v_k,$$

respectively, for $x \in \mathfrak{M}$ and $v_i \in \mathfrak{M}^{\otimes n}$.

Corollary 6.7. *We have two infinite commutative diagrams*

$$\begin{array}{ccccccc}
\dots & J_{2n+1}(G) & \xrightarrow{\widetilde{\iota}'_1} & J_{n+1}(G) & \xrightarrow{\widetilde{\iota}'_0} & J_1(G) \\
 & \downarrow & & \downarrow & & \downarrow \\
\dots & J_{2n}(G) & \xrightarrow{\widetilde{\iota}_1} & J_n(G) & \xrightarrow{\widetilde{\iota}_0} & J_0(G)
\end{array}$$

where in one diagram the vertical arrows are $\iota_{nk} : J_{nk+1}(G) \longrightarrow J_{nk}(G)$*, and in the other they are* $p_{nk} : J_{nk+1}(G) \longrightarrow J_{nk}(G)$*.*

The inverse limit of the spaces J_{nk} *with respect to the maps* $\widetilde{\iota}_k$ *and the inverse limit of the spaces* J_{nk+1} *with respect to the maps* $\widetilde{\iota}'_k$ *are homeomorphic to the limit space* $\mathcal{J}_G$*. If we identify these limits with each other by the limit of the maps* ι_{nk}*, then the limit of the maps* p_{nk} *is a dynamical system topologically conjugate to* $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$*.*

Proof. The homeomorphism of the inverse limit of the maps $\widetilde{\iota}_k$ with $\mathcal{J}_G$ constructed in the proof of Theorem 5.9 maps the point of the inverse limit represented by a sequence

$$(\xi_1 \otimes v_1, \quad \xi_2 \otimes v_2 \otimes v_1, \quad \xi_3 \otimes v_3 \otimes v_2 \otimes v_1, \ldots),$$

where $\xi_{k+1} = \widetilde{I}(\xi_k \otimes v_k)$, to the point of $\mathcal{J}_G$ represented by $\cdots \otimes v_3 \otimes v_2 \otimes v_1 \in \mathcal{X}_G$. Taking tensor product of the spaces Γ_{nk} with $\mathfrak{M}$, and using the identification $\mathcal{X}_G \cong \mathcal{X}_G \otimes \mathfrak{M}$, we get a natural homeomorphism of the limit of the spaces Γ_{nk+1} with the space $\mathcal{X}_G$, mapping

$$(\xi_1 \otimes v_1 \otimes x, \quad \xi_2 \otimes v_2 \otimes v_1 \otimes x, \quad \xi_3 \otimes v_3 \otimes v_2 \otimes v_1 \otimes x, \ldots)$$

to $\cdots \otimes v_3 \otimes v_2 \otimes v_1 \otimes x$. It is easily checked now that if we make these identifications of the inverse limits with $\mathcal{J}_G$, then the limit of the maps ι_{nk} will be identical on $\mathcal{J}_G$, and the limit of the maps p_{nk} will be the shift $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$. □

7. Examples of contracting topological automata

7.1. **A self-covering of the torus.** Consider the self-similar group G generated by

$$u = \sigma(1, uv), \quad v = \sigma(u^{-1}, v).$$

It is checked directly that u and v commute and that they have infinite order. Consequently, this group is isomorphic to the free abelian group $\mathbb{Z}^2$, and one can apply the general theory (see [Nek05, Section 2.9]) to check that it is contracting, and to find the limit dynamical system. It is conjugate to the self-covering of the torus $\mathbb{C}/\mathbb{Z}[i]$ induced by multiplication by $(1 - i)$ on $\mathbb{C}$. Nevertheless, let us apply in this simple setting the cut-and-paste procedure described in Proposition 6.5 and find a simplicial approximation of the limit dynamical system.

The nucleus of the group generated by u and v is the set

$$\mathcal{N} = \{1, u, v, u^{-1}, v^{-1}, uv, u^{-1}v^{-1}\},$$

see [Nek08b]. Let us find the adjacency sets $A \subset \mathcal{N}$ containing the identity. The set of maximal simplices of the Rips complex $\Gamma(G, \mathcal{N})$ containing the identity is

$$\mathcal{A} = \{\{1, u, uv\}, \{1, v, uv\}, \{1, u^{-1}, u^{-1}v^{-1}\}, \{1, v^{-1}, u^{-1}v^{-1}\}, \{1, u, v^{-1}\}, \{1, u^{-1}, v\}\}.$$

We have

$$\begin{aligned}
\{1, u, uv\}|_0 &= \{1, v\}, & \{1, u, uv\}_1 &= \{1, uv, v\}, \\
\{1, v, uv\}|_0 &= \{1, u^{-1}, v\}, & \{1, v, uv\}|_1 &= \{1, v\} \\
\{1, u^{-1}, u^{-1}v^{-1}\}|_0 &= \{1, v^{-1}, u^{-1}v^{-1}\}, & \{1, u^{-1}, u^{-1}v^{-1}\}|_1 &= \{1, v^{-1}\}, \\
\{1, v^{-1}, u^{-1}v^{-1}\}|_0 &= \{1, v^{-1}\}, & \{1, v^{-1}, u^{-1}v^{-1}\}|_1 &= \{1, u, v^{-1}\}, \\
\{1, u, v^{-1}\}|_0 &= \{1, v^{-1}\}, & \{1, u, v^{-1}\}|_1 &= \{1, u, uv\}, \\
\{1, u^{-1}, v\}|_0 &= \{1, u^{-1}, u^{-1}v^{-1}\}, & \{1, u^{-1}, v\}|_1 &= \{1, v\}.
\end{aligned}$$

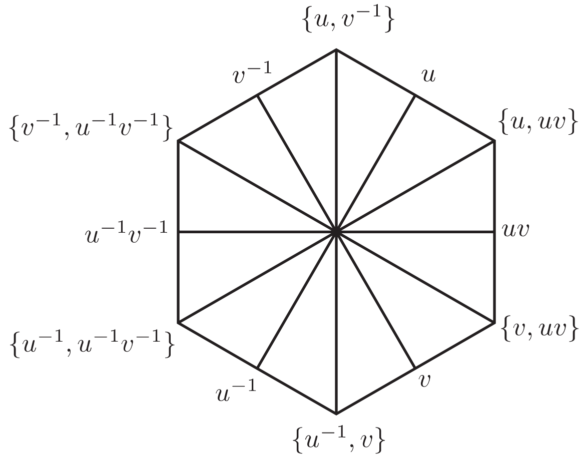

FIGURE 2. Complex T_0

It follows from Corollary 6.2 that all the elements of $\mathcal{A}$ are adjacency sets. The geometric realization T_0 of the poset of adjacency sets containing the identity is shown on Figure 7.1. (The identity element is not listed in the labels, e.g., a label g denotes the vertex corresponding to $\{1, g\}$.)

For every $g \in \mathcal{N} \setminus \{1\}$ the set $K_{g,0}$ is the side of the hexagon containing the vertex corresponding to $\{1, g\}$. The transformation $\kappa_{g,0}$ identifies the opposite sides $K_{g,0}$ and $K_{g^{-1},0}$ of the hexagon. We conclude that the complex $J_0(G)$ is a two-dimensional torus.

The dual Moore diagram of the nucleus is shown on the left-hand part of Figure 7.1. On the right-hand side of the figure the complex T_1 is shown. It follows from the dual Moore diagram and Proposition 6.5 that T_1 is obtained by gluing two copies of T_0 along two edges (containing $\{1, u\}$ in the copy $(T_0, 0)$ and containing $\{1, u^{-1}\}$ in $(T_0, 1)$).

The labels inside the hexagons on Figure 7.1 describe the covering map $p : J_1(G) \longrightarrow J_0(G)$, i.e., they repeat the labels of T_0 in its copies (T_0, x). The labels outside describe the map $\iota : J_1(G) \longrightarrow J_0(G)$. The highlighted vertices are mapped to vertices of the hexagon T_0. The labels show to which sides of the hexagon T_0 the corresponding edges of T_1 are mapped (in particular, a letter $g \in \mathcal{N}$ labels the edges of the domains $K_{g,1}$ of the identifications $\kappa_{g,1}$, described in Proposition 6.5).

The simplicial map ι maps the two highlighted portions of T_1 to single vertices ($\{1, v\}$ and $\{1, v^{-1}\}$, respectively): it maps the top half of the top hexagon and the bottom half of the bottom hexagon to the top and the bottom halves of the hexagon T_0, respectively; the remaining part of T_1 is mapped to the horizontal axis of symmetry of T_0 (passing through $\{1, uv\}$ and $\{1, u^{-1}v^{-1}\}$. See Figure 7.1, where the complex T_6 is shown, which was obtained by application of Proposition 6.5. The hexagon T_0 is superimposed with T_6 in such a way that the vertices of the hexagon T_0 coincide with their preimages under ι^6.

It is easy to see that the map $\iota : J_1(G) \longrightarrow J_0(G)$ is homotopic to a homeomorphism $\widetilde{\iota}$. When we replace ι by $\widetilde{\iota}$, we will transform the topological nucleus of G into a subdivision rule $(J_0(G), J_1(G), p, \widetilde{\iota})$ defining a self-covering $p : J_1(G) \longrightarrow J_0(G)$ of a torus (where $J_1(G)$ and $J_0(G)$ are identified with each other by $\widetilde{\iota}$). For instance,

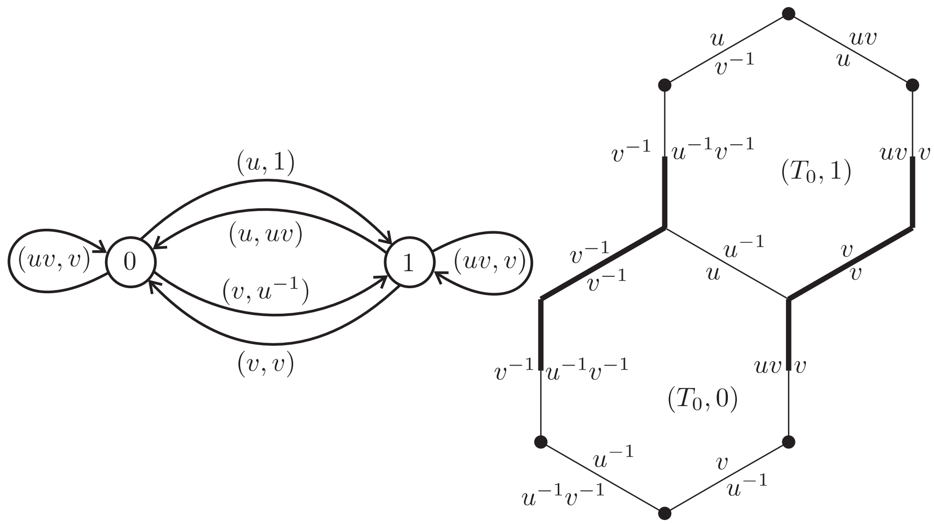

FIGURE 3. The dual Moore diagram of the nucleus and T_1

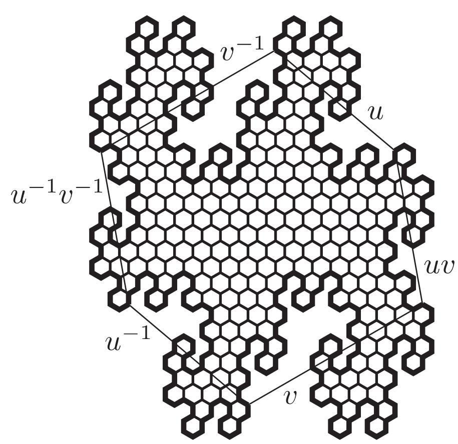

FIGURE 4. The complex T_6

by computing the action of the self-covering p on the homology, we conclude that it is homotopic to the self-covering of $\mathbb{C}/\mathbb{Z}[i]$ induced by $z \mapsto (1-i)z$.

7.2. **A Fornæss-Sibony example.** The following rational transformation of $\mathbb{C}^2$ was studied in [FS92].

$$f(z,p) = \left(\left(1-\frac{2z}{p}\right)^2, \quad \left(1-\frac{2}{p}\right)^2\right).$$

The map f can be extended to an endomorphism of $\mathbb{CP}^2$. The post-critical set of f is then the union of the lines $z=1, z=0, p=1, p=0$, $p=z$ and the line at infinity.

The iterated monodromy group of f was computed in [Nek08a]. It is easier to describe the iterated monodromy group of the quotient of the map f by the complex conjugation $(z, p) \mapsto (\overline{z}, \overline{p})$, which will be an index two extension of $\mathrm{IMG}(f)$. It is the group G generated by the transformations

$$\begin{aligned} \alpha &= \sigma, & a &= \pi, \\ \beta &= (\alpha, \gamma, \alpha, \gamma), & b &= (a\alpha, a\alpha, c, c), \\ \gamma &= (\beta, 1, 1, \beta), & c &= (b\beta, b\beta, b, b), \end{aligned}$$

where $\sigma = (12)(34)$ and $\pi = (13)(24)$. The iterated monodromy group of f is generated then by $\alpha, \beta, \gamma, S = ac\gamma$ and $T = cb$.

Direct computation shows that the nucleus of G is a union of the following six finite groups

$$\begin{aligned} G_A = &\quad \langle \beta, \gamma, b, c\rangle &\cong D_8 \rtimes D_4, \\ G_B = &\quad \langle \alpha, \gamma, a, c\rangle &\cong D_4 \rtimes D_2, \\ G_C = &\quad \langle \alpha, \beta, a, b\rangle &\cong D_8 \rtimes D_4, \end{aligned}$$

and

$$\begin{aligned} G_{A_1} = &\quad \langle \alpha, b, c\rangle &\cong C_2 \times D_4, \\ G_{B_1} = &\quad \langle \beta, a, c\gamma\rangle &\cong C_2 \times D_2, \\ G_{C_1} = &\quad \langle \gamma, a\alpha, b\beta\rangle &\cong C_2 \times D_4, \end{aligned}$$

where D_n denotes the dihedral group of order $2n$ and C_n is a cyclic group of order n. Note that the group of inner automorphisms of D_{2n} is isomorphic to D_n, which defines the corresponding semidirect products above.

Inspection of the Moore diagram of the nucleus shows that these six subgroups G_* are precisely the maximal adjacency sets containing the identity element.

Consider the poset $\mathfrak{G}$ of the subgroups G_* and their all possible intersections (pairwise and triple are enough, since all the rest are trivial). One can show that for every $H \in \mathfrak{G}$ and every $x \in \{1, 2, 3, 4\}$, the set $H|_x$ is also an element of $\mathfrak{G}$. It follows that the set $\overline{\Gamma}$ of cosets $H \cdot g$ for $H \in \mathfrak{G}$ and $g \in G$ is a G-invariant subcomplex of the barycentric subdivision Γ' of the tiling nerve Γ of G, and that $\overline{\Gamma}$ is invariant under the maps $I_x : \Gamma' \longrightarrow \Gamma'$.

It follows that restricting the equivariant map $I : \Gamma \otimes \mathfrak{M} \longrightarrow \Gamma$ onto $\overline{\Gamma} \otimes \mathfrak{M}$ we get a combinatorial model $(\overline{J}_0(G), \overline{J}_1(G), p, \iota)$ of f.

The complex $\overline{J}_0(G)$ is the geometric realization of the poset $\mathfrak{G}$. It is a union of three tetrahedra with a common face.

The recursive definition of the complexes $\overline{J}_n(G)$, approximating the limit space of G is a simple pasting rule, which has a nice interpretation in the spirit of Hubbard trees. The Julia set of f is approximated then by two copies of $\overline{J}_n(G)$ glued together in a natural way. See for more details the paper [Nek08a].

7.3. **Post-critically finite rational functions.** Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a post-critically finite complex rational function. Let $\mathcal{M}$ be the Thurston orbifold of f and let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be the associated topological automaton. The underlying space of the orbifold $\mathcal{M}$ is a punctured sphere if f has a super-attracting cycle (i.e., a cycle containing a critical point) and is the whole sphere, if every critical point is strictly pre-periodic. In all these cases f is expanding with respect to the Poincaré metric on $\mathcal{M}$. There exists a compact subset $\mathcal{M}' \subset \mathcal{M}$ such that

it contains all singular points of $\mathcal{M}$, and $f^{-1}(\mathcal{M}') \subset \mathcal{M}'$ (one can take $\mathcal{M}'$ to be the set bounded by appropriate level curves of the Green function of the Julia set of f). Restricting the Poincaré metric onto $\mathcal{M}'$ we get a contracting topological automaton $\mathcal{F}' = (\mathcal{M}', \mathcal{M}'_1, f, \iota)$.

Proposition 7.1. *If f has a super-attracting cycle (in particular, when it is a polynomial), then there exists n such that $f^{\circ n}$ is combinatorially equivalent to a contracting automaton $\mathcal{F} = (J, J_1, f, \iota)$, where J is a graph of cyclic groups.*

Proof. If f has a super-attracting cycle, then the orbifold $\mathcal{M}'$ can be retracted to a graph of groups, and we can use Proposition 5.12. □

In many cases, choosing a nice retract of the Thurston orbifold and choosing a correct metric on the retract, one can find a contracting combinatorial model of f, and not just of $f^{\circ n}$ for some n. Such constructions are classical for post-critically finite polynomials.

7.4. **Hubbard trees of strictly pre-periodic polynomials.** Let f be a post-critically finite polynomial such that every finite critical point of f is strictly pre-periodic (i.e., has finite forward orbit, but does not belong to a cycle). Then the polynomial has no attracting cycles in $\mathbb{C}$, therefore its Julia set J_f is a dendrite, i.e., every two points $x, y \in J_f$ can be connected by a unique arc. The set $P_f \setminus \{\infty\}$ of finite post-critical points of f is a subset of J_f. The *Hubbard tree* of f is the convex hull of the set of finite post-critical points in J_f. Here the convex hull of a set $A \subset J_f$ is the union of the arcs connecting all pairs of points of A. The Hubbard tree is a natural choice for a retract of the Thurston orbifold.

The Hubbard tree H_f is invariant, i.e., $f(H_f) = H_f$. For every point $x \in J_f$ there exists a unique point $y \in H_f$ such that the arc connecting x with y has no common points with H_f except for y. The point y is called *projection* of x onto H_f. In particular, projection of a point $x \in H_f$ onto H_f is the point x itself. It is not hard to show that the projection map $\iota : J_f \longrightarrow H_f$ is continuous.

Denote by $\mathcal{M}$ the orbispace with the underlying space H_f, with the orbispace structure obtained by restricting the Thurston orbispace of f onto H_f (see the definition of the Thurston orbispace in Subsection 3.2.8). Let $\mathcal{M}_1$ be the orbispace with the underlying space $f^{-1}(H_f)$ such that $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a $\deg(f)$-fold covering of orbispaces. Then the projection map $\iota : f^{-1}(H_f) \longrightarrow H_f$ is a morphism of the orbispaces.

The iterate $\mathcal{M}_n$ of the automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ is homeomorphic to the convex hull of the set $f^{-n}(P_f)$ in the Julia set J_f.

The obtained topological automaton is contracting with respect to an appropriate metric on H_f. It is combinatorially equivalent to the polynomial f (i.e., to the corresponding partial self-coverings) by Proposition 4.7.

Consequently, the Hubbard tree is a model of the dynamical system (J_f, f), by Theorem 5.10. Hubbard trees are used extensively in symbolic dynamics of polynomial iterations, see [DH84, DH85, BS02].

As an example consider the polynomial $z^2 + i$. The orbit of its critical value i is $i \mapsto -1 + i \mapsto -i \mapsto -1 + i$. Figure 5 shows on its left-hand side the Julia set of $z^2 + i$. On the right-hand side of the figure the Hubbard trees $\mathcal{M}$ and $\mathcal{M}_1$ are shown. Black dots mark the singular points of the corresponding orbispaces. Isotropy group of each of the singular points is of order 2. The point 0 is critical (but non-singular). The morphism $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ maps the whole branch of $\mathcal{M}_1$

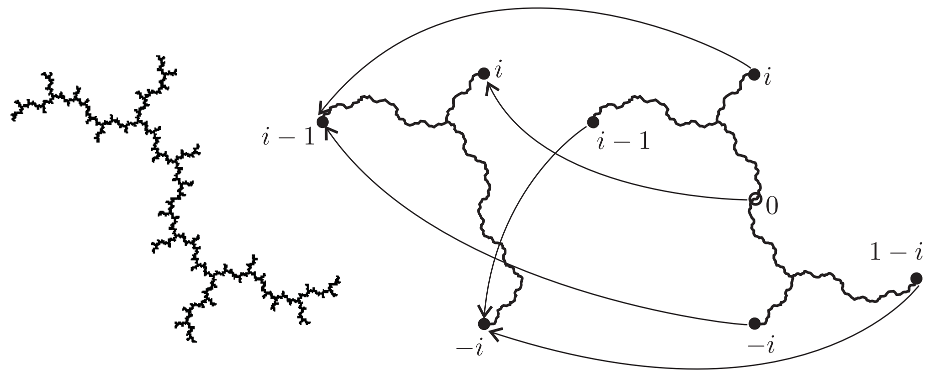

FIGURE 5. Hubbard tree of $z^2 + i$

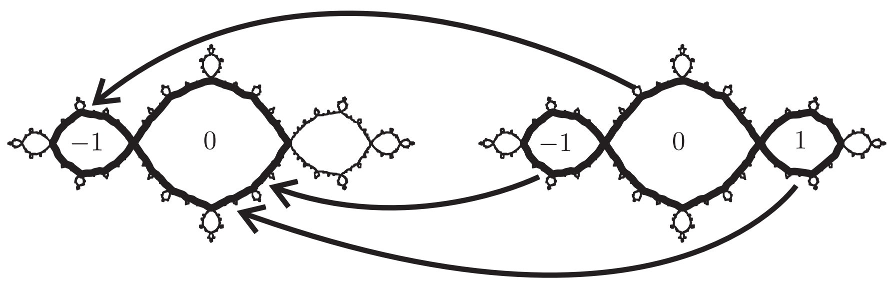

FIGURE 6. Basilica

containing $1 - i$ to the point at which this branch is connected to $\mathcal{M}$, and "kills" the isotropy group. It acts identically on the rest of the Hubbard tree.

7.5. **Hubbard graphs of polynomials.** Hubbard trees can be also defined for arbitrary post-critically finite polynomials, see [DH84, DH85]. A more appropriate construction in our setting is a modification of the classical construction of Hubbard trees, obtained by replacing some of its vertices by circles. Instead of formulating a general construction, we will just describe two examples.

7.5.1. *Basilica.* Consider the polynomial $f(z) = z^2 - 1$. Its post-critical set is $\{0, -1, \infty\}$. The Julia set of $z^2 - 1$, called *Basilica*, is shown on Figure 6. Let $\mathcal{M}$ be the union of the boundaries of the Fatou components containing the finite post-critical points 0 and -1 (it is highlighted on the left-hand side part of the figure). The set $\mathcal{M}$ is homotopically equivalent to $\widehat{\mathbb{C}} \setminus \{0, -1, \infty\}$ and is forward invariant. Let $\mathcal{M}_1 = f^{-1}(\mathcal{M})$. It is the union of the boundaries of the Fatou components of $0, 1$ and -1 (highlighted on the right-hand side part of Figure 6). The arrows on Figure 6 show the action of f.

It is easy to see now that the topological automaton

$$\mathcal{F} = (\mathbb{C} \setminus \{0, -1\}, \mathbb{C} \setminus \{0, 1, -1\}, f, id)$$

is homotopically equivalent to the automaton $\mathcal{F}_1 = (\mathcal{M}, \mathcal{M}_1, f, \iota)$, where ι is identical on $\mathcal{M} \subset \mathcal{M}_1$ and maps the boundary of the Fatou component of 1 to its common point with $\mathcal{M}$.

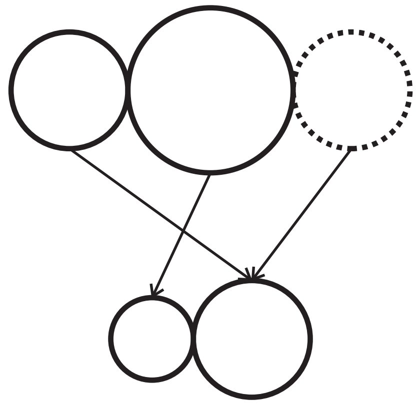

FIGURE 7. Affine model of Basilica

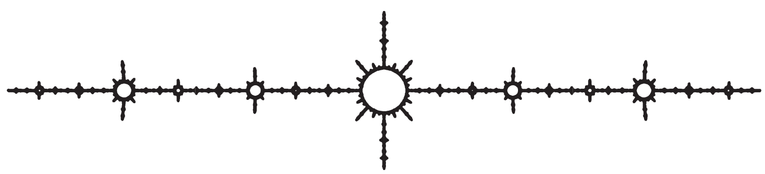

FIGURE 8. Airplane

The space $\mathcal{M}_n$ obtained by iteration of the constructed automaton is homeomorphic to the union of the boundaries of the Fatou components containing the points of the set $f^{-n}(\{0, -1\})$.

The fact that the automaton $\mathcal{F}_1$ is contracting can be shown directly by introduction of a natural length structure on $\mathcal{M}$ and $\mathcal{M}_1$, and considering an abstract affine model of $\mathcal{F}_1$. The space $\mathcal{M}$ will be a one point union of a circle of length 1 and a circle of length $\sqrt{2}$. Let $\mathcal{M}_1$ be the double locally isometric covering of $\mathcal{M}$ such that the circle of length 1 is doubly covered by a circle of length 2, and the circle of length $\sqrt{2}$ is covered by two isometric circles. See the covering on Figure 7.

Let $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be the continuous map contracting one of the two f-preimages of the circle of length $\sqrt{2}$ (shown by a dashed line on Figure 7) to its common point with the circle of length 2, dividing by $\sqrt{2}$ all the distances in the other two circles of $\mathcal{M}_1$ and then mapping them isometrically onto $\mathcal{M}$. The obtained automaton is topologically conjugate to $\mathcal{F}_1$.

7.5.2. *Airplane.* Figure 8 shows the Julia set the "Airplane" polynomial $z^2 + c$ for $c \approx -1.7549\ldots$, which is determined by the condition that it has real coefficients, and the critical point 0 belongs to a cycle of length three. We can use again the boundaries of the Fatou components of the post-critical points to construct a simple contracting topological automaton combinatorially equivalent to the polynomial. The main difference with the case of the polynomial $z^2 - 1$ is that these boundaries are disjoint.

Hence, one has to attach the circles corresponding to the boundaries to each other imitating their relative arrangement in the Julia set. The corresponding topological automaton is shown on Figure 9.

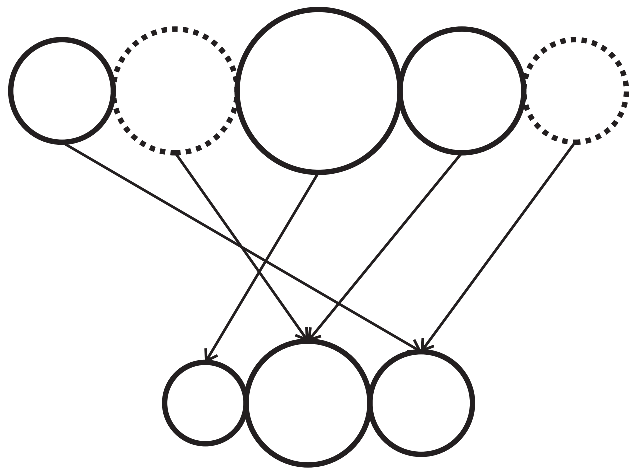

Figure 9. Model of the Airplane

The circles of $\mathcal{M}$ are of lengths 1, $\sqrt[3]{4}$, $\sqrt[3]{2}$; the space $\mathcal{M}_1$ covers the circle of the length 1 twice by a circle of length 2, and covers the remaining two circles isometrically by pairs of circles. The dashed lines show the circles, which are collapsed by the map ι. It divides the lengths of the other circles by $\sqrt[3]{2}$.

7.6. **Correspondences on moduli spaces.** Let $f : S^2 \longrightarrow S^2$ be a Thurston map (i.e., an orientation preserving post-critically finite branched self-covering of the sphere) of degree d. Let P_f be the post-critical set of f.

We present here a short summary of the Teichmüller theory of Thurston maps. For more details and for relation of these concepts with a theorem of Thurston, see [DH93].

The *Teichmüller space* $\mathcal{T}_{P_f}$ modelled on (S^2, P_f) is the space of homeomorphisms $\tau : S^2 \longrightarrow \widehat{\mathbb{C}}$ (seen as complex structures on S^2), where two complex structures $\tau_1, \tau_2 : S^2 \longrightarrow \widehat{\mathbb{C}}$ are identified if there exists a Möbius transformation $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ such that $\phi \circ \tau_1$ is isotopic to τ_2 relative to P_f (and is equal to τ_2 on P_f).

For every complex structure $\tau \in \mathcal{T}_{P_f}$ there exists a unique complex structure $\tau' \in \mathcal{T}_{P_f}$, such that the map $f_\tau = \tau \circ f \circ (\tau')^{-1}$ closing the commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^2 \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} \\ \widehat{\mathbb{C}} & \xrightarrow{f_\tau} & \widehat{\mathbb{C}} \end{array} \tag{9}$$

is a rational function. Let us denote $\tau' = \sigma_f(\tau)$.

The *moduli space* $\mathcal{M} = \mathcal{M}_{P_f}$ of (S^2, P_f) is the space of injective maps $P_f \longrightarrow \widehat{\mathbb{C}}$ modulo compositions with Möbius transformations. It is known that $\mathcal{T}_{P_f}$ is the universal covering of $\mathcal{M}_{P_f}$, where the covering map is $\tau \mapsto \tau|_{P_f}$.

The fundamental group of $\mathcal{M}_{P_f}$ can be identified with the (pure) mapping class group G of (S^2, P_f), so that the action of the fundamental group on the universal covering $\mathcal{T}_{P_f}$ coincides with the action of the mapping class group on $\mathcal{T}_{P_f}$ by

compositions with the homeomorphisms:

$$g : \tau \mapsto \tau \circ g$$

for $g \in G$ and $\tau \in \mathcal{T}_{P_f}$.

Every homeomorphism $g \in G$ can be lifted by the Thurston map $f : S^2 \longrightarrow S^2$ to a homeomorphism of S^2, which we will denote by f^*g. Let G_1 be the subgroup of the elements $g \in G$ such that f^*g fixes the set $P_f \subset f^{-1}(P_f)$ pointwise, i.e., is an element of G. It is easy to see that G_1 is a subgroup of finite index in G. The map $g \mapsto f^*g$ is a homomorphism from G_1 to G, i.e., it is a virtual endomorphism of the group G.

Let $\mathcal{M}_1$ be the quotient of $\mathcal{T}_{P_f}$ by the action of G_1. Then the identity map on $\mathcal{T}_{P_f}$ induces a finite covering $F : \mathcal{M}_1 \longrightarrow \mathcal{M}$.

On the other hand, since $f^{-1}(P_f) \supseteq P_f$, we get a well defined continuous map

$$\iota : \mathcal{M}_1 \longrightarrow \mathcal{M} : \tau \mapsto \sigma_f(\tau)|_{P_f}.$$

We will call the obtained topological automaton $(\mathcal{M}, \mathcal{M}_1, F, \iota)$ the *moduli space correspondence* associated with the Thurston map f. It is easy to check that the virtual endomorphism $g \mapsto f^*g$ is associated with the topological automaton $(\mathcal{M}, \mathcal{M}_1, F, \iota)$.

As before, we interpret the topological automaton as the correspondence $\iota(x) \mapsto f(x)$, which in this case is the projection of the correspondence $\sigma_f(\tau) \mapsto \tau$ onto the moduli space.

In some cases (but not in general) ι is one-to-one and we get hence partial self-covering of $\mathcal{M}$.

As an example consider a quadratic polynomial $f(z) = z^2 + c$ such that the critical point 0 belongs to a cycle of length n, and look at it just as at a Thurston map $f : S^2 \longrightarrow S^2$. Let $\infty, 0, c = z_0, \ldots, z_{n-2}$ be the post-critical set P_f, where $z_k = f(z_{k-1})$ and $0 = f(z_{n-2})$. Let $\tau : S^2 \longrightarrow \widehat{\mathbb{C}}$ be an arbitrary point of the Teichmüller space $\mathcal{T}_{P_f}$. The corresponding point of the moduli space $\mathcal{M}$ is determined by the values of $\tau(\infty)$ and $\tau(z_k)$ for $k = 0, \ldots, n-2$.

Applying an appropriate Möbius transformation, we may assume that $\tau(\infty) = \infty$, $\tau(0) = 0$ and $\tau(z_0) = 1$. It follows that the corresponding point of the moduli space is determined by the tuple

$$(\tau(z_1), \tau(z_2), \ldots, \tau(z_{n-2})) = (p_1, p_2, \ldots, p_{n-2}) \in \mathbb{C}^{n-2}.$$

In this way we identify the moduli space $\mathcal{M}$ with the set

$$\{(p_1, p_2, \ldots, p_{n-2}) \in \mathbb{C}^{n-2} \ : \ p_i \neq 0, p_i \neq 1, p_i \neq p_j \text{ for } i \neq j\}.$$

Let $(p'_1, p'_2, \ldots, p'_{n-2}) \in \mathcal{M}$ be the point of the moduli space corresponding to $\sigma_f(\tau)$. Then it follows from the commutative diagram (9), that f_τ is a quadratic polynomial with critical point 0 such that

$$f_\tau(0) = 1, f_\tau(1) = p'_1, f_\tau(p_1) = p'_2, \ldots, f_\tau(p_{n-3}) = p'_{n-2}, f_\tau(p_{n-2}) = 0.$$

The first equality and the fact that 0 is a critical point imply that $f_\tau(z) = 1 + az^2$ for some $a \in \mathbb{C}$. It follows from the equality $f_\tau(p_{n-2}) = 0$ that $a = -\frac{1}{p_{n-2}^2}$. The remaining equalities imply

$$(p'_1, p'_2, \ldots, p'_{n-2}) = \left(1 - \frac{1}{p_{n-2}^2}, 1 - \frac{p_1^2}{p_{n-2}^2}, \ldots, 1 - \frac{p_{n-3}^2}{p_{n-2}^2}\right).$$

It follows that the correspondence $\sigma_f(\tau) \mapsto \tau$ is projected onto the moduli space to the rational function $(p_1, \ldots, p_{n-2}) \mapsto (p'_1, \ldots, p'_{n-2})$, so that the moduli space correspondence is a partial self-covering.

Note that in this case the rational map can be extended to an endomorphism of $\mathbb{CP}^{n-2}$ given in homogeneous coordinates by

$$[p_1 : p_2 : \cdots : p_{n-2} : p_{n-1}] \mapsto [p_{n-2}^2 - p_{n-1}^2 : p_{n-2}^2 - p_1^2 : \cdots : p_{n-2}^2 - p_{n-3}^2 : p_{n-2}^2].$$

The post-critical set of this endomorphism is the union of the lines $p_i = 0, p_i = 1, p_i = p_j$ for all $i = 1, \ldots, n-1$ and $i \neq j = 1, \ldots, n-1$.

For more on this and similar post-critically finite endomorphisms of complex projective spaces, see [Koc07].

Let us describe a combinatorial model of the moduli space correspondence for the case when f is a topological polynomial. Here a Thurston map $f : S^2 \longrightarrow S^2$ is called a topological polynomial if there exists a point $x \in S^2$ such that $f^{-1}(x) = \{x\}$. In this case x is called the *point at infinity*, and f is considered to be a branched self-covering of the plane $S^2 \setminus \{x\}$.

Our combinatorial model will be a topological automaton (D_n, D'_n, p, ι), where D_n and D'_n are affine polyhedra, the covering p is a local isometry and ι is contracting, provided the topological polynomial f is *hyperbolic*, i.e., every cycle of f in P_f contains a critical point. The space D_n is the moduli space of a family of planar graphs, thus its definition is similar to the construction of the classifying space of the outer automorphism group of the free group, given in [CV86] by M. Culler and K. Vogtmann. The complex D_n is also closely related to the classifying space of the braid groups defined in [Bra01].

The polyhedron D_n will depend only on the size of the post-critical set P_f. Let $|P_f| = n+1$ so that f has n finite post-critical points. A *cactus diagram of n discs* is an oriented two-dimensional contractible cellular complex Γ consisting of n discs labeled by numbers from 1 to n, such that any two disc are either disjoint or have only one common point on their boundaries. A *planar cactus diagram* is a cactus diagram together with an isotopy class of an orientation preserving embedding $\Delta : \Gamma \longrightarrow \mathbb{R}^2$ into the plane. The isotopy class is uniquely determined by the cyclic orders of the discs adjacent to every given disc of the diagram.

A *metric cactus diagram* is a cactus diagram together with a metric on the one-skeleton of the diagram, such that perimeter of a disc labeled by k is equal to a fixed positive number l_k. A *planar metric diagram* is a metric cactus diagram together with an isotopy class of an orientation preserving embedding into the plane.

The cells of the polyhedron D_n are in a bijective correspondence with the planar cactus diagrams of n discs, while the points of D_n are in a bijective correspondence with metric planar cactus diagrams. Points of a given cell are obtained by specifying the lengths of the edges in the one-skeleton of the corresponding diagram, so that the perimeters of the discs are equal to the chosen numbers l_k. It follows that dimension of a cell is equal to the number of the vertices of the corresponding diagram minus one. When some of the distances go to zero, the number of vertices of the planar diagram decreases and the corresponding point of D_n approaches to a cell of lower dimension.

In particular, the polyhedron D_n has $(n-1)!$ vertices, corresponding to planar diagrams in which all discs have one common point (a bouquet of discs). There are no distances to specify. One-dimensional edges of D_n correspond to diagrams with two vertices, so that we have to specify one distance. The maximal number of

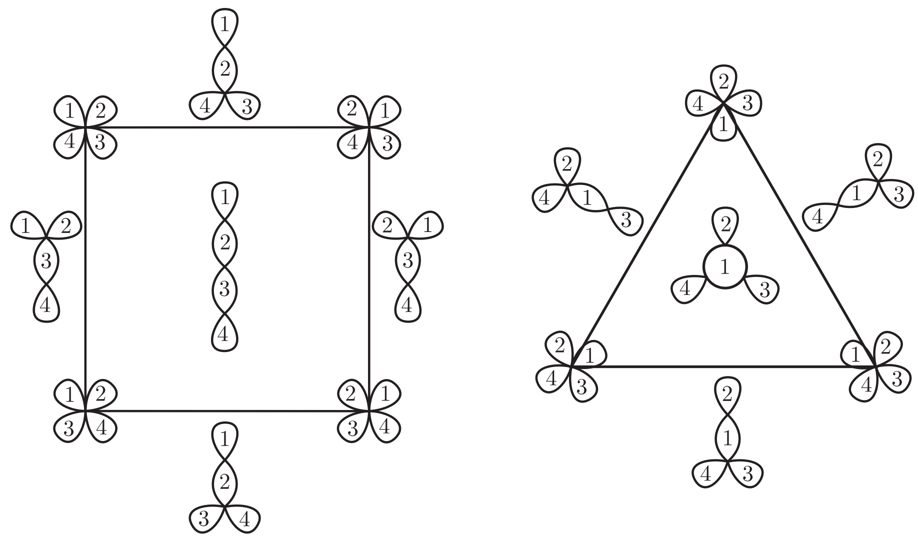

FIGURE 10. Cells of D_4

vertices for a cactus diagram of n discs is $n-1$, so the polyhedron D_n is $(n-2)$-dimensional.

We use the lengths of the edges of the one-skeleta of the diagrams as affine coordinates on the corresponding cell. For a given planar diagram, the set of possible metric realizations (i.e., the corresponding cell) is a direct product of simplices, due to the constrains on the perimeters of each of the discs.

As an example, see Figure 10, where 2-cells of D_4 are described. The complex D_3 is shown on the left-hand side of Figure 11.

Let now $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a topological polynomial with n post-critical points $z_1, \ldots, z_n$.

For every metric planar diagram $\Delta \in D_n$ find a representative $\widetilde{\Delta} : \Gamma \longrightarrow \mathbb{R}^2$ of the corresponding isotopy class such that the image of the disc labeled by k contains in its interior the point z_k for every $k = 1, \ldots, n$. Let $f^{-1}(\widetilde{\Delta})$ be the lift of $\widetilde{\Delta}$ by f. It is a planar diagram of $|f^{-1}(\{z_1, \ldots, z_n\})|$ discs with one preimage of a post-critical point in each disc. We introduce a metric on this planar diagram by lifting it from Δ. We have $\{z_1, \ldots, z_n\} \subset f^{-1}(\{z_1, \ldots, z_n\})$, so that all post-critical points belong to interiors of some of the discs of the diagram $f^{-1}(\widetilde{\Delta})$. For each given Δ there is only a finite number of possibilities for the isotopy class of $f^{-1}(\widetilde{\Delta})$ and for the assignments of the post-critical points to the discs of $f^{-1}(\widetilde{\Delta})$. We get in this way a finite number of metric planar diagrams $f^{-1}(\widetilde{\Delta})$ in which some discs are labeled by post-critical points z_k. The space D'_n of such diagrams is also an affine polyhedron, such that the map $p : f^{-1}(\widetilde{\Delta}) \mapsto \Delta$ is an isometric covering.

In each of the labeled diagrams $f^{-1}(\widetilde{\Delta})$ contract the non-labeled discs to points and rescale the perimeters of the remaining labeled discs so that the disc containing the point z_k has perimeter l_k. We will get in this way a metric planar diagram $\iota(f^{-1}(\widetilde{\Delta})) \in D_n$ (we label the disc containing z_k by k). We have defined in this way a piecewise affine map $\iota : D'_n \longrightarrow D_n$.

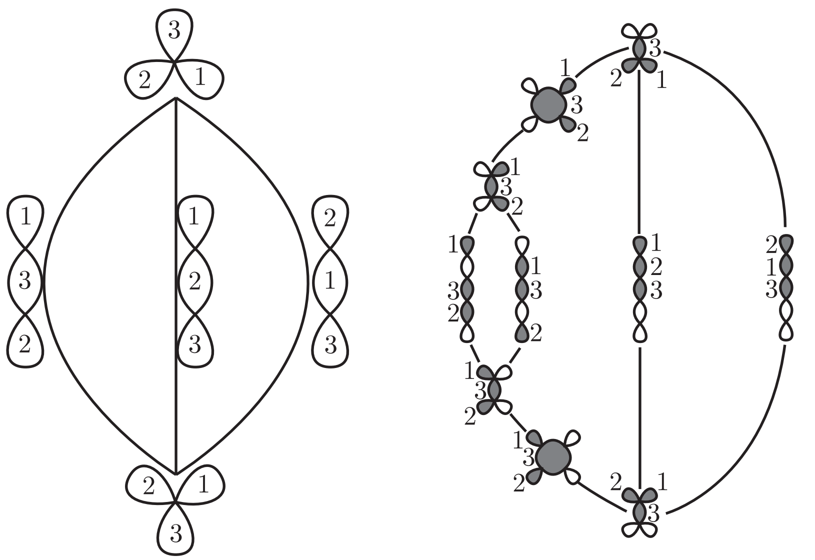

FIGURE 11. A moduli space model of $1 - 1/z^2$

Theorem 7.2. *The topological automaton* $\mathcal{F} = (D_n, D'_n, p, \iota)$ *associated with a post-critically finite polynomial* f *is combinatorially equivalent to the moduli correspondence associated with* f.

If the polynomial f *is hyperbolic then the automaton* $\mathcal{F}$ *is contracting.*

Proof. Let $P = \{z_i\}_{i=1,\ldots,n}$ be, as above, the set of finite post-critical points of f. Consider the space $\widetilde{D}_n$ of isotopy classes relative to P of embeddings $\Delta : \Gamma \longrightarrow \mathbb{R}^n$ of metric cactus diagrams of n discs such that the image of the disc number k contains the point z_k in its interior.

The pure mapping class group G of $(S^2, P \cup \infty)$ acts naturally on $\widetilde{D}_n$. The action is free (in particular, since the action of the mapping class group on the fundamental group of $\mathbb{R}^2 \setminus P$ by outer automorphism group is faithful). The quotient of $\widetilde{D}_n$ by the action is the space D_n, hence the action is co-compact.

For an embedding $\Delta : \Gamma \longrightarrow \mathbb{R}^n$, representing a point of $\widetilde{D}_n$, consider the lift of Δ by f, contract in the lift the discs that do not contain points of P in their interior, and rescale the perimeters of the remaining discs accordingly to the indices of post-critical points contained in them. We will get then a point $\Phi(\Delta)$ of $\widetilde{D}_n$. The map Φ obviously satisfies the condition

$$\Phi(\Delta \cdot g) = \Phi(\Delta) \cdot (f^* g),$$

for all elements $g \in G_1$. Here $\Delta \cdot g$ is the image of Δ under the action of g, and $G_1 \leq G$ is the subgroup of elements of G lifted to elements of G by the branched covering f (see above). It follows that Φ agrees with the virtual endomorphism associated with the moduli correspondence $(\mathcal{M}, \mathcal{M}_1, F, \iota)$ associated with f, hence the associated topological automaton $\mathcal{D} = (\widetilde{D}_n/G, \widetilde{D}_n/G_1, P, \varphi)$, where $P : \widetilde{D}_n/G_1 \longrightarrow \widetilde{D}_n/G$ is the covering induced by the inclusion $G_1 < G$, and φ is the map induced by Φ, is combinatorially equivalent to the moduli correspondence.

It follows directly from the definitions that the topological automaton $\mathcal{D}$ is isomorphic to (D_n, D'_n, p, ι). It is also easy to show that if every cycle of f contains a critical point, then Φ is contracting. □

Figure 11 shows the complexes D_3 and D'_3 for a quadratic polynomial f such that its finite critical point belongs to a cycle of length three. The labels $1, 2, 3$ correspond to the post-critical points z_1, z_2, z_3, where $f(z_k) = z_{k+1}$ and z_3 is the critical point. On the right hand side of Figure 11 we show the diagrams $f^{-1}(\widetilde{\Delta})$, where grey cells are the cells containing post-critical points.

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